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THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_n OVER p-ADIC FIELDS

The aim of the seminar is to understand Henniart's formulation and proof of the Local Langlands Correspondence for GL_n over p-adic local fields, using the formalism of L- and ϵ-factors of pairs. One of the formulations of this correspondence is the following theorem.

Theorem. *Let K be a finite extension of $\mathbb{Q}_p$. Then there is a unique family of bijections, indexed by the positive integers n, between the sets of irreducible representations σ of the absolute Galois group G_K of K of dimension n and supercuspidal irreducible smooth representations π of $\mathrm{GL}_n(K)$ whose central character has finite order, denoted $\sigma \mapsto \pi(\sigma)$, satisfying the following compatibilities:*

- *if $n = 1$, then this bijection is induced by local class field theory;*
- *in general, the central character of $\pi(\sigma)$ corresponds to $\det \sigma$ by local class field theory;*
- *it is compatible with twisting: For any character χ of G_K, $\pi(\sigma \otimes \chi) = \pi(\sigma) \otimes (\pi(\chi) \circ \det)$;*
- *it is compatible with contragredients: $\pi(\sigma^\vee) = \pi(\sigma)^\vee$;*
- *it preserves L- and ϵ-factors of pairs: Fix a nontrivial additive character ψ of K. Then for any σ_1, σ_2,*

$$L(\pi(\sigma_1) \times \pi(\sigma_2), s) = L(\sigma_1 \otimes \sigma_2, s)$$
$$\epsilon(\pi(\sigma_1) \times \pi(\sigma_2), s, \psi) = \epsilon(\sigma_1 \otimes \sigma_2, s, \psi) \ .$$

A large part of the seminar will be devoted to thoroughly understanding the formulation of this theorem, in particular the definition of L- and ϵ-factors of pairs. We remark that this is the natural 'nonabelian' generalization of local class field theory, and that it is well-known how to extend this bijection first to a bijection between all supercuspidal irreducible smooth representations of $\mathrm{GL}_n(K)$ and irreducible n-dimensional representations of the Weil group W_K of K and then to a bijection between *all* irreducible smooth representations of $\mathrm{GL}_n(K)$ and Frobenius-semisimple n-dimensional representations of the Weil-Deligne group WD_K of K.

The corresponding statement for local fields of characteristic p was proved by Laumon-Rapoport-Stuhler in [11].

There is a second proof of the Local Langlands Correspondence due to Harris-Taylor, [6]. Their proof also shows that the local correspondence is compatible with a global correspondence in many cases. However, their proof of the local correspondence itself is much closer to Henniart's proof than suggested by most survey articles. General introductions to the Local Langlands Correspondence and its proof include [14], where a very detailed discussion of both sides of the correspondence is given, and [3], where more emphasis is put on the proofs. For a broader perspective of the Langlands program, there is, among many others, the book containing the articles [10], [4], and the book [1].

Although the statement of the Local Langlands Correspondence is purely local, all known proofs (for general n)[1] make heavy use of global arguments. The aim of the seminar is to see the constant interplay between local and global arguments.

[1] For $n = 1$, i.e. local class field theory, there is a local proof. More recently, there is a complete local proof for $n = 2$, [2]. For $n > 2$, there are only partial results.

1. Talk. L-functions on the Galois side
Explain the functional equation for the L-function of complex representations of the Weil group of number fields F, as in [12]. In particular, recall the necessary background on local and global Weil groups and their relationship, prove Proposition 2.3.1, and define local L-factors and L-functions of complex representations of the Weil group. Only show the existence of some global ϵ-factor, the more precise statement that the 'global' ϵ-factor is a product of 'local' ϵ-factors will be proved in the next talk. The functional equation for L-functions of Hecke characters should be considered known. The statement $H^2(G_F, \mathbb{C}^\times) = 1$ (used in (2.2.3)) can be left without proof.

2. Talk. ϵ-factors on the Galois side
Give Deligne's proof of the existence of ϵ-factors for representations of the Weil groups W_K of local (non-archimedean) fields K, [5], Section 4. Explain the stability of ϵ-factors under twisting by very ramified characters which is a crucial ingredient in the proof. The same proof is also explained in [13] for representations of the Galois group.

3. Talk. Modular forms and cuspidal automorphic representations for $\mathrm{GL}_2/\mathbb{Q}$
Explain how to convert classical modular forms into automorphic forms on $\mathrm{GL}_2/\mathbb{Q}$, as in [10], Sections 1 and 2. Explain as much as possible on how to deduce the equivalence between systems of Hecke eigenvalues arising from cuspidal eigenforms of weight k and cuspidal automorphic representations π of $\mathrm{GL}_2/\mathbb{Q}$ whose component π_∞ is the holomorphic discrete series representation of weight k, cf. [10], Section 4.

4. Talk. Cuspidal automorphic representations for GL_n: Multiplicity 1
Explain the general definition of a cuspidal automorphic representation for GL_n as in [1], Section 3.3. State the theorems on the complete decomposition of L^2_0, the tensor product theorem, the existence of global Whittaker models and local uniqueness of Whittaker models and indicate proofs where possible (following the general policy of the book [1] to restrict to $n = 2$ and $\mathbb{Q}$ whenever this helps). Deduce some form of (strong) multiplicity 1; state the general form (i.e., without the restriction at archimedean places).

5. Talk. L-functions and ϵ-factors on the automorphic side
Explain the definition of the global integrals defining L-functions of pairs of cuspidal automorphic representations as in [4], at least for the case of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$. Explain the factorization of the global integrals into local integrals and state the main theorems on the theory of the local integrals to define L- and ϵ-factors for pairs of generic representations, cf. also [14]. Explain how this yields a global L-function with the desired functional equation. It would be nice to see enough of the theory for $\mathrm{GL}_n \times \mathrm{GL}_n$ to understand the proof of the strong multiplicity 1 theorem explained in [4]. In any case, include as many proofs as possible.

6. Talk. Uniqueness of the Local Langlands Correspondence
Give the proof of the main result of the article [8], in the formulation with generic representations; also state the version with supercuspidal representations as a corollary without proof.

7. Talk. Equality of L- and ϵ-factors almost everywhere implies equality of pairs everywhere
Combine and streamline the arguments from [9] and [7] to prove Corollary 2.4 in [9]. At some point, one needs the classification of unitary generic representations of $\mathrm{GL}_n(K)$, due to Tadić: Give a complete statement of this result, but leave it without proof. We also accept all needed facts on L- and ϵ-factors on the automorphic side without proof. The rest of the proof should be given in some detail.

8. Talk. Reduction to the construction of sufficiently many cuspidal automorphic representations
Give a detailed proof of the construction of an *injective* map $\sigma \mapsto \pi(\sigma)$ satisfying all the desired properties of the Local Langlands Correspondence, assuming Theorem 3.1 in [9], i.e. explain Section 2 and 4 of [9]. Explain that the numerical Local Langlands Correspondence, also proved by Henniart, then implies that this is a bijection.

9. Talk. Harris' non-Galois automorphic induction
Explain Harris' construction of certain cuspidal automorphic representations, cf. e.g. [6], Section 7.2 and [9], Section 3. This is extremely technical. The basic idea is however very simple, cf. e.g. [3], Section 3.5.

References

[1] D. Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[2] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for* GL(2), volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[3] H. Carayol. Preuve de la conjecture de Langlands locale pour GL_n: travaux de Harris-Taylor et Henniart. *Astérisque*, (266):Exp. No. 857, 4, 191–243, 2000. Séminaire Bourbaki, Vol. 1998/99.

[4] J. W. Cogdell. Analytic theory of L-functions for GL_n. In *An introduction to the Langlands program (Jerusalem, 2001)*, pages 197–228. Birkhäuser Boston, Boston, MA, 2003.

[5] P. Deligne. Les constantes des équations fonctionnelles des fonctions L. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 501–597. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973.

[6] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[7] G. Henniart. On the local Langlands conjecture for $\mathrm{GL}(n)$: the cyclic case. *Ann. of Math. (2)*, 123(1):145–203, 1986.

[8] G. Henniart. Caractérisation de la correspondance de Langlands locale par les facteurs ϵ de paires. *Invent. Math.*, 113(2):339–350, 1993.

[9] G. Henniart. Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps p-adique. *Invent. Math.*, 139(2):439–455, 2000.

[10] S. S. Kudla. From modular forms to automorphic representations. In *An introduction to the Langlands program (Jerusalem, 2001)*, pages 133–151. Birkhäuser Boston, Boston, MA, 2003.

[11] G. Laumon, M. Rapoport, and U. Stuhler. $\mathcal{D}$-elliptic sheaves and the Langlands correspondence. *Invent. Math.*, 113(2):217–338, 1993.

[12] J. Tate. Number theoretic background. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979.

[13] J. T. Tate. Local constants. In *Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 89–131. Academic Press, London, 1977. Prepared in collaboration with C. J. Bushnell and M. J. Taylor.

[14] T. Wedhorn. The local Langlands correspondence for $\mathrm{GL}(n)$ over p-adic fields. In *School on Automorphic Forms on* $\mathrm{GL}(n)$, volume 21 of *ICTP Lect. Notes*, pages 237–320. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2008.