Prof. Dr. Valentin Blomer
Dr. Edgar Assing

Relative Trace Formulae in Analytic Number Theory
Solution to Set 1

Remark: *We start by briefly outlining the approach using Poincaré series. Afterwards we will set up the relative trace formula approach using an appropriate Bergman kernel (i.e. the reproducing kernel for the space of cuspidal Jacobi Forms). Note that the Bergman kernel arises also in the adelic framework as matrix coefficient of the corresponding discrete series representation. For the sake of this solution setting the adelic picture up properly would take us to far afield. Therefore we will stay in the classical world.*

A quick look at Poincaré series: For $k \geq 3$ there are functions $P_{k,m,(r,n)} \in J_{k,m}^{\mathrm{cusp}}$ called Poincaré series that are uniquely determined by the property

$$\langle f, P_{k,m,(n,r)} \rangle = \alpha_{k,m} \cdot (4mn - r^2)^{-k+\frac{3}{2}} \cdot c_f(n,r) \text{ for } \alpha_{k,m} = \frac{m^{k-2}\Gamma(k-\frac{3}{2})}{2\pi^{k-\frac{3}{2}}}$$

and all $f \in J_{k,m}^{\mathrm{cusp}}$. We can write them down as

$$P_{k,m,(n,r)}(\tau,z) = \sum_{\Gamma_\infty^J \backslash \Gamma^J} [e^{n,r}|_{k,m}\gamma](\tau,z),$$

where we write $e^{n,r}(\tau,z) = e(n\tau + rz)$. The assumption $k \geq 3$ ensures that this sum converges absolutely and uniformly on compact sets. It therefore clearly defines a function in $J_{k,m}^{\mathrm{cusp}}$. The desired property is checked using a typical unfolding argument:

$$\begin{aligned}\langle f, P_{k,m,(n,r)} \rangle &= \int_{\Gamma_\infty^J \backslash (\mathbb{H}\times\mathbb{C})} f(\tau,z)\overline{e(n\tau+rz)}\mu_{k,m}^2 dV \\ &= \sum_{n',r'} c_f(n',r') \int_0^\infty \int_0^1 \int_{\mathbb{R}} \int_0^1 e((n'-n)u + (r'-r)x) e^{-2\pi((n'+n)v + (r'+r)y} \\ &\qquad \cdot v^{k-3} e^{-4\pi m\frac{y^2}{v}} dudvdxdy \\ &= c(n,r) \int_0^\infty e^{-4\pi n v} v^{k-3} \left(\int_{\mathbb{R}} e^{-4\pi(ry+my^2/v)} dy \right) dv.\end{aligned}$$

The remaining integrals are easily evaluated and should give the correct constants. We write the Fourier expansion of the Poincaré series as

$$P_{k,m,(n,r)}(\tau,z) = \sum_{\substack{n',r' \in \mathbb{Z}, \\ 4mn'-r'^2>0}} g_{k,m,(n,r)}(n',r') e(n'\tau + r'z).$$

We can now compute

$$\begin{aligned}\alpha_{k,m} \cdot (4mn - r^2)^{-k+\frac{3}{2}} \cdot g_{k,m,(n',r')}(n,r) &= \langle P_{k,m,(n',r')}, P_{k,m,(n,r)} \rangle \\ &= \sum_{f \in \mathrm{ONB}(J_{k,m}^{\mathrm{cusp}})} \langle P_{k,m,(n',r')}, f \rangle \langle f, P_{k,m,(n,r)} \rangle \\ &= \alpha_{k,m}^2 (4mn - r^2)^{-k+\frac{3}{2}} (4mn' - r'^2)^{-k+\frac{3}{2}} \sum_{f \in \mathrm{ONB}(J_{k,m}^{\mathrm{cusp}})} c_f(n,r)\overline{c_f(n',r')}.\end{aligned}$$

The desired Petersson formula once we can compute the Fourier coefficients of Poincaré series. This is done as follows. We parametrize $\Gamma_\infty^J\backslash\Gamma^J$ by triples $\lambda, c, d \in \mathbb{Z}$ with $(c,d)=1$ via

$$(c,d,\lambda) \leftrightsquigarrow \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda a, \lambda b), 1\right),$$

for some choice of $a, b \in \mathbb{Z}$ with $ad - bc = 1$. We get

$$P_{k,m,(n,r)}(\tau,z) = \sum_{\substack{c,d,\lambda\in\mathbb{Z},\\(c,d)=1}} (c\tau+d)^{-k} e\left(\frac{-cz^2}{c\tau+d} + \lambda^2\frac{a\tau+b}{c\tau+d} + 2\lambda\frac{z}{c\tau+d}\right)^m \cdot e\left(\frac{a\tau+b}{c\tau+d}\right)^n e\left(\frac{z}{c\tau+d} + \lambda\frac{a\tau+b}{c\tau+d}\right)^r.$$

We compute the contribution of $c=0$ first. In this case we have $d=\pm1$ and get a contribution of

$$\sum_{\lambda\in\mathbb{Z}} e((\lambda^2 m + r\lambda + n)\tau) \sum_{\pm} (\pm1)^k e(\pm(2m\lambda + r)z) = \sum_{\substack{n',r'\in\mathbb{Z},\\4mn'-r'^2>0}} [\delta(n,r,n',r') + (-1)^k\delta(n,r,n',-r')]e(n'\tau + r'z),$$

with

$$\delta(n,r,n',r') = \begin{cases} 1 & \text{if } r'^2 - 4mn' = r^2 - 4mn \text{ and } r' \equiv r \bmod 2m, \\ 0 & \text{else.} \end{cases}$$

We need to compute the terms for $c \neq 0$. Note that if $c < 0$, we can replace z by $-z$ and reduce to the case $c > 0$ at the price of $(-1)^k$. Keeping this in mind we assume that $c > 0$. We will need the identities

$$\begin{aligned} &\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1}{c(c\tau+d)}, \\ &\frac{z}{c\tau+d} + \lambda\frac{a\tau+b}{c\tau+d} = \frac{z-\lambda/c}{c\tau+d} + \lambda\frac{a}{c} \text{ and} \\ &\lambda^2\frac{a\tau+b}{c\tau+d} + 2\lambda\frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d} = -\frac{c(z-\lambda/c)^2}{c\tau+d} + \lambda^2\frac{a}{c}. \end{aligned}$$

We keep the sum over $c > 0$ as it is, but we split the d- and λ-sum into congruence classes modulo c. After a little computation (with the identities above) this leads to

$$\sum_{c\geq1} c^{-k} \sum_{\substack{d \bmod c,\\(d,c)=1}} \sum_{\lambda \bmod c} e(\frac{(m\lambda^2 + r\lambda + n)\overline{d}}{c}) F_{k,m,c,(n,r)}(\tau + \frac{d}{c}, z - \frac{\lambda}{c}),$$

where

$$F_{k,m,c,(n,r)}(\tau,z) = \sum_{\alpha,\beta\in\mathbb{Z}} (\tau+\alpha)^{-k} e\left(-\frac{(z-\beta)^2}{\tau+\alpha}\right)^m e\left(\frac{-1}{c^2(\tau+\alpha)}\right)^n e\left(\frac{z-\beta}{c(\tau+\alpha)}\right)^r.$$

We compute the Fourier expansion of this function using Poisson summation:

$$F_{k,m,c,(n,r)}(\tau, z) = \sum_{n',r' \in \mathbb{Z}} \gamma(n', r') e(n'\tau + r'z)$$

with

$$\gamma(n', r') = \int_{(c_1)} \tau^{-k} e(-n'\tau) \int_{(c_2)} e(-\frac{m}{\tau} z^2 + \frac{rz}{c\tau} - \frac{n}{c^2\tau} - r'z) dz d\tau.$$

The inner integral can be simplified to

$$e(-\frac{rr'}{2mc}) e(\frac{D'}{4m}\tau + \frac{D}{4mc^2}\tau^{-1}) \underbrace{\int_{(c_2')} e(-\frac{m}{\tau} z^2) dz}_{=(\frac{\tau}{2im})^{\frac{1}{2}}} \text{ for } D' = r'^2 - 4mn' \text{ and } D = r^2 - 4mn$$

using the substitution $z \to z + \frac{1}{2m}(\frac{r}{c} - r'\tau)$. We arrive at

$$\gamma(n', r') = (2m)^{-\frac{1}{2}} e(-\frac{rr'}{2mc}) \int_{(c_1)} (\tau/i)^{\frac{1}{2}} \tau^{-k} e(\frac{D'}{4m}\tau + \frac{D}{4mc^2}\tau^{-1}) d\tau.$$

After an easy substitution the remaining integral can be realised as an inverse Laplace transform:

$$\begin{aligned}\gamma(n', r') &= \frac{2\pi}{\sqrt{2m}} e(-\frac{rr'}{2mc}) i^{-k} c^{k-\frac{3}{2}} (D'/D)^{\frac{k}{2}-\frac{3}{4}} \cdot \frac{1}{2\pi i} \int_{(c_1')} s^{-k+\frac{1}{2}} e^{\frac{2\pi}{4mc}\sqrt{D'D}(s-s^{-1})} ds \\ &= \frac{2\pi}{\sqrt{2m}} e(-\frac{rr'}{2mc}) i^{-k} c^{k-\frac{3}{2}} (D'/D)^{\frac{k}{2}-\frac{3}{4}} \cdot J_{k-\frac{3}{2}}(\frac{\pi}{mc}\sqrt{D'D}).\end{aligned}$$

(The integral is easy to find in the literature.) Collecting everything together yields

$$\begin{aligned} g_{k,m,(n,r)}(n', r') &= \delta(n, r, n', r') + (-1)^k \delta(n, r, n', -r') \\ &\quad + \frac{i^k \sqrt{2}\pi}{\sqrt{m}} (D'/D)^{\frac{k}{2}-\frac{3}{4}} \sum_{c \geq 1} \sum_{\pm} (\pm 1)^k H_{m,c}(n, r, n', \pm r') J_{k-\frac{3}{2}}(\frac{\pi}{cm}\sqrt{DD'}),\end{aligned}$$

where additional to the notation introduced throughout the computation we have

$$H_{m,c}(n, r, n', r') = c^{-\frac{3}{2}} \sum_{\substack{\rho \bmod c, \\ (\rho,c)=1}} \sum_{\lambda \bmod c} e\left(\frac{(m\lambda^2 + r\lambda + n)\overline{\rho} + n'\rho - r'\lambda}{c} - \frac{rr'}{2mc}\right).$$

The final formula turns out to be

$$\begin{aligned}\alpha_{k,m} \left((4mn - r^2)(4mn' - r'^2)\right)^{-\frac{k}{2}+\frac{3}{4}} \sum_{f \in \mathrm{ONB}(J_{k,m}^{\mathrm{cusp}})} c_f(n, r)\overline{c_f(n', r')} &= \delta(n', r', n, r) + (-1)^k \delta(n', r', n, -r) \\ + \frac{i^k \sqrt{2}\pi}{\sqrt{m}} \sum_{\pm} (\pm 1)^k \sum_{c \geq 1} H_{m,c}(n', r', n, \pm r) J_{k-\frac{3}{2}}(\frac{\pi}{mc}\sqrt{(4mn - r^2)(4mn' - r'^2)})\end{aligned}$$

as expected.

The relative trace formula approach: We define

$$h_{k,m}(\tau,z,\tau_0,z_0)=(\tau-\overline{\tau_0})^{-k}e\left(-m\frac{(z-\overline{z_0})^2}{\tau-\overline{\tau_0}}\right).$$

We take the following properties for granted:[1]

- The function $h_{k,m}$ is invariant under the action $|_{k,m}\times\overline{|_{k,m}}$ of $\mathfrak{J}(\mathbb{R})$.
- For $k>3$ we have
$$\int_{\mathbb{H}\times\mathbb{C}}|h_{k,m}(\tau,z,\tau_0,z_0)|\mu_{k,m}(\tau,z)dV<\infty.$$
- For a holomorphic function $f\colon\mathbb{F}\times\mathbb{C}\to\mathbb{C}$ so that $f\mu_{k,m}$ is bounded we have
$$\int_{\mathbb{H}\times\mathbb{C}}f(\tau,z)\overline{h_{k,m}(\tau,z,\tau_0,z_0)}\mu_{k,m}(\tau,z)dV=\frac{2^{2-k}\pi i^k}{(2k-3)m}f(\tau_0,z_0).$$

We write $\lambda_{k,m}=\frac{(2k-3)m}{2^{2-k}\pi i^k}$ and denote by $|_{k,m}^{(1)}$ the slash operator applied to the first variable. Then we define

$$K(\tau,z,\tau_0,z_0)=\lambda_{k,m}\sum_{\gamma\in\Gamma^J}[h_{k,m}|_{k,m}^{(1)}\gamma](\tau,z,\tau_0,z_0).$$

It can be shown that for each (τ_0,z_0) the series is absolutely and uniformly convergent on compacta. Further, this function is the reproducing kernel for the Hilbert space $J_{k,m}^{\mathrm{cusp}}$. This means that for all $(\tau_0,z_0)\in\mathbb{H}\times\mathbb{C}$ we have $K(*,\tau_0,z_0)\in J_{k,m}^{\mathrm{cusp}}$ and

$$\langle f,K(*,\tau_0,z_0)\rangle=f(\tau_0,z_0)\text{ for all }f\in J_{k,m}^{\mathrm{cusp}}.$$

We get the alternative description

$$K(\tau,z,\tau_0,z_0)=\sum_{f\in\mathrm{ONB}(J_{k,m}^{\mathrm{cusp}})}f(\tau,z)\overline{f(\tau_0,z_0)}.$$

To get our relative trace formula we integrate

$$I(n,r,n',r')=\int_0^1\int_0^1\int_0^1\int_0^1K(\tau,z,\tau_0,z_0)e(-nu-rx)\overline{e(-n'u_0-r'x_0)}dudxdu_0dx_0.$$

Using the spectral expansion we find

$$I(n,r,n',r')=e^{-2\pi(nv+ry+n'v_0+r'y_0)}\sum_{f\in\mathrm{ONB}(J_{k,m}^{\mathrm{cusp}})}c_f(n,r)\overline{c_f(n'r')}.$$

It remains to compute the geometric side. This can be done by writing

$$I(n,r,n',r')=\lambda_{k,m}\sum_{\gamma\in\Gamma_\infty^J\backslash\Gamma^J}\int_{[0,1]^2}\int_{\mathbb{R}^2}[h_{k,m}|_{k,m}^{(1)}\gamma](\tau,z,\tau_0,z_0)$$
$$\cdot\, e(-nu-rx)\overline{e(-n'u_0-r'x_0)}dudxdu_0dx_0$$

[1] Checking these is actually a very pleasing undertaken which we skip.

We can parametrize the set $\Gamma^J_\infty\backslash\Gamma^J$ as before by triples (c,d,λ) of integers with $(c,d)=1$. For fixed c we denote the contribution from this c by I_c. We first compute I_0:

$$I_0=\lambda_{k,m}\sum_{\pm}(\pm 1)^k\int_{[0,1]^2}\int_{\mathbb{R}^2}\sum_{\lambda\in\mathbb{Z}}e(\lambda^2\tau\pm 2\lambda z)^m\cdot(\tau\pm b-\overline{\tau_0})^{-k}e(-m\cdot\frac{(z\pm\lambda\tau+\lambda b-\overline{z_0})^2}{\tau\pm b-\overline{\tau_0}})$$
$$\cdot e(-nu-rx+n'u_0+r'x_0)du_0dx_0dudx.$$

This should be of course independent of our choice for b and indeed this is seen by a simple change in the u_0 and x_0 variables. We get

$$I_0=\lambda_{k,m}\sum_{\pm}(\pm 1)^k\int_{[0,1]^2}\int_{\mathbb{R}^2}\sum_{\lambda\in\mathbb{Z}}e(\lambda^2\tau\pm 2\lambda z)^m\cdot(\tau-\overline{\tau_0})^{-k}e(-m\cdot\frac{(z\pm\lambda\tau-\overline{z_0})^2}{\tau-\overline{\tau_0}})$$
$$\cdot e(-nu-rx+n'u_0+r'x_0)du_0dx_0dudx.$$

Further changes of variables yield

$$I_0=\lambda_{k,m}\sum_{\pm}(\pm 1)^k\int_{\mathbb{R}^2}\sum_{\lambda\in\mathbb{Z}}(iv-\overline{\tau_0})^{-k}e(-m\cdot\frac{(iy\pm\lambda iv-\overline{z_0})^2}{iv-\overline{\tau_0}})e(m\lambda^2 iv\pm 2m\lambda iy+n'u_0+r'x_0)$$
$$\cdot\int_{[0,1]^2}\cdot e((m\lambda^2+n'\pm\lambda r'-n)u+(r'\pm 2m\lambda-r)x)dudxdu_0dx_0.$$

Evaluating the inner integrals gives

$$m\lambda^2\pm\lambda r'=n-n' \text{ and } \pm 2m\lambda=r-r'.$$

Which precisely picks up $\delta(n,\pm r,n',r')$ as in the Poincaré series approach. Furthermore λ is determined uniquely. We get

$$I_0=\lambda_{k,m}\sum_{\pm}(\pm 1)^k\delta(n,\pm r,n',r')\int_{\mathbb{R}^2}(iv-\overline{\tau_0})^{-k}e(-m\cdot\frac{(iy\pm\lambda iv-\overline{z_0})^2}{iv-\overline{\tau_0}})e(m\lambda^2 iv\pm 2m\lambda iy+n'u_0+r'x_0).$$

Putting $a=v+v_0$ and $b=y\pm\lambda v+y_0$ we need to evaluate the integrals

$$\mathcal{I}=\int_{\mathbb{R}}(u_0+ia)^{-k}e(-n'u_0)\int_{\mathbb{R}}e(-m\frac{(x_0+ib)^2}{u_0+ia}-r'x_0)dx_0du_0.$$

The inner integral is standard.[2] Indeed we get

$$\int_{\mathbb{R}}e(-m\frac{(x_0+ib)^2}{u_0+ia}-r'x_0)dx_0=e(ir'b)\int_{\mathbb{R}}e(-m\frac{(x_0+ib)^2}{u_0+ia}-r'(x_0+ib))dx_0$$
$$=\sqrt{-i(u_0+ia)}(2m)^{-\frac{1}{2}}e(ir'b+\frac{r'^2}{4m}(u_0+ia)).$$

[2]For $\alpha,\beta\in\mathbb{C}$ with $\Im(\alpha)>0$ one has

$$\int_{\mathbb{R}}e(\alpha(x_0+ib)^2+\beta(x_0+ib))dx_0=\sqrt{\frac{i}{2\alpha}}e(-\frac{\beta^2}{4\alpha}).$$

This leads to

$$\mathcal{I} = e(in'a + ir'b)\sqrt{\frac{-i}{2m}} \int_{\mathbb{R}} (u_0 + ia)^{-k+\frac{1}{2}} e((\frac{r'^2}{4m} - n')(u_0 + ia)du_0.$$

This is easily identified as a Γ-function. We end up with

$$\mathcal{I} = e(in'a + ir'b)\frac{\sqrt{2}\pi i^k}{\sqrt{m}} \cdot \frac{1}{2\pi i}\int_{(a')} z^{-(k-\frac{3}{2})} e^{\pi \frac{D'}{2m} z}\frac{dz}{z}.$$

$$= \delta_{D'>0} e(in'a + ir'b) \cdot \frac{\pi^{k-\frac{1}{2}} i^k}{2^{k-2} m^{k-1} \Gamma(k - \frac{1}{2})} (D')^{k-\frac{3}{2}}.$$

Gathering everything we find

$$I_0 = e(m\lambda^2 iv \pm 2m\lambda iy + in'a + ir'b) \cdot \frac{\lambda_{k,m}\pi^{k-\frac{1}{2}} i^k}{2^{k-2} m^{k-1}\Gamma(k-\frac{1}{2})}(D')^{k-\frac{3}{2}} \sum_{\pm} (\pm)^k \delta(n, r, n', \pm r')$$

$$= e^{-2\pi(nv+ry+n'v_0+r'y_0)} \frac{2\pi^{k-\frac{3}{2}}}{m^{k-2}\Gamma(k-\frac{3}{2})}(D')^{k-\frac{3}{2}} \sum_{\pm} (\pm 1)^k \delta(n, r, n', \pm r').$$

This should be exactly what we expect, (after recalling $\alpha_{k,m}$). We turn towards the case $c > 0$. (The case $c < 0$ can be treated similarly and we omit the details for now.) We take

$$\gamma = \gamma_{d,\lambda} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} [\lambda a, \lambda b]$$

and write

$$j_{k,m}(\gamma, \tau, z) = (c\tau + d)^{-k} e\left(-\frac{cz^2}{c\tau + d} + \lambda^2 \frac{a^\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d}\right)^m$$

$$= (c\tau + d)^{-k} e\left(-\frac{c(z - \lambda/c)^2}{c\tau + d} + \lambda^2 \frac{a}{c}\right)^m.$$

We need to compute

$$I_c = \lambda_{k,m} \sum_{(d,c)=1} \sum_{\lambda} \int_{[0,1]^2} \int_{\mathbb{R}^2} j_{k,m}(\gamma_{d,\lambda}, \tau, z) \left(\frac{a\tau + b}{c\tau + d} - \overline{\tau_0}\right)^{-k} e(-m\frac{(\frac{z}{c\tau+d} + \lambda\frac{a\tau+b}{c\tau+d} - \overline{z}_0)^2}{\frac{a\tau+b}{c\tau+d} - \overline{\tau_0}})$$
$$\cdot e(-nu - rx)\overline{e(-n'u_0 - r'x_0)} dudxdu_0dx_0$$

We split the d and the λ-sum in congruence classes modulo c and make some changes of variables to obtain

$$I_c = (-c)^k \lambda_{k,m} \sum_{\substack{d \bmod c, \\ (d,c)=1}} \sum_{\lambda \bmod c} e\left((m\lambda^2 + n' + \lambda r')\frac{\overline{d}}{c} + \frac{nd}{c} - r\frac{\lambda}{c}\right)$$
$$\cdot \int_{\mathbb{R}^4} \left(1 + c^2\overline{\tau_0}\tau\right)^{-k} \cdot e(-mc\frac{cz^2\overline{\tau_0} - c\tau\overline{z_0}^2 + 2z\overline{z_0}}{1 + c^2\tau\overline{\tau_0}})$$
$$\cdot e(-nu - rx + n'u_0 + r'x_0)dudxdu_0dx_0.$$

This nicely identifies the correct Kloosterman sum $H_{m,c}(n',r',n,r)$ up to the normalizing factors. Computing the remaining integrals is a bit tricky and we only sketch the argument. We start by computing x and x_0 integrals. This is essentially a double Gaussian integral and one obtains

$$\int_{\mathbb{R}^2} e(-mc\frac{cz^2\overline{\tau_0} - c\tau\overline{z_0}^2 + 2z\overline{z_0}}{1+c^2\tau\overline{\tau_0}}) \cdot e(-rx + r'x_0)dxdx_0$$
$$= e^{-2\pi(ry+r'y_0)}\frac{\sqrt{1+c^2\tau\overline{\tau_0}}}{2mc} e\left(\frac{1}{4mc}(r^2c\tau - 2rr' - r'^2c\overline{\tau_0})\right).$$

Inserting this above gives

$$I_c = e^{-2\pi(nv+ry+n'v_0+r'y_0)}\frac{(-c)^{k+\frac{1}{2}}}{2mi}\lambda_{k,m}H_{m,c}(n,-r,n',r')$$
$$\cdot \underbrace{\int_{\mathbb{R}^2}\left(1+c^2\overline{\tau_0}\tau\right)^{\frac{1}{2}-k} \cdot e\left(\frac{1}{4mc}(r^2c\tau - r'^2c\overline{\tau_0}) - n\tau + n'\overline{\tau_0}\right) dudu_0}_{=\mathcal{I}'}.$$

The remaining integral is easily transformed to

$$\begin{aligned}
\mathcal{I}' &= \int_{\mathbb{R}}(c^2\overline{\tau_0}^2)^{\frac{1}{2}-k}e(-\frac{D'}{4m}\tau_0)\int_{\mathbb{R}}\left(c^{-2}\overline{\tau_0}^{-1}+\tau\right)^{\frac{1}{2}-k}\cdot e\left(\frac{D}{4mc}\tau\right)dudu_0\\
&= \int_{\mathbb{R}}(c^2\overline{\tau_0})^{\frac{1}{2}-k}e(-\frac{D'}{4m}\overline{\tau_0} - \frac{D}{4mc^2}\overline{\tau_0}^{-1})\int_{\mathbb{R}}\tau^{-(k-\frac{3}{2})}\cdot e\left(\frac{D}{4mc}\tau\right)\frac{du}{\tau}du_0\\
&= c^{1-2k}\frac{i^{\frac{1}{2}-k}\pi^{k-\frac{1}{2}}D^{k-\frac{3}{2}}}{m^{k-\frac{3}{2}}2^{k-\frac{5}{2}}\Gamma(k-\frac{1}{2})}\int_{\mathbb{R}}\overline{\tau_0}^{\frac{3}{2}-k}e(-\frac{D'}{4m}\overline{\tau_0} - \frac{D}{4mc^2}\overline{\tau_0}^{-1})\frac{du_0}{\overline{\tau_0}}\\
&= c^{-\frac{1}{2}-k}(DD')^{\frac{k}{2}-\frac{3}{4}}\frac{i^{\frac{1}{2}-k}\pi^{k-\frac{1}{2}}}{m^{k-\frac{3}{2}}2^{k-\frac{5}{2}}\Gamma(k-\frac{1}{2})}\int_{\mathbb{R}}\overline{\tau_0}^{\frac{3}{2}-k}e\left(-\frac{\sqrt{DD'}}{4mc}(\overline{\tau_0}+\overline{\tau_0}^{-1})\right)\frac{du_0}{\overline{\tau_0}}\\
&= c^{-\frac{1}{2}-k}(DD')^{\frac{k}{2}-\frac{3}{4}}\frac{(-1)^k\pi^{k+\frac{1}{2}}}{m^{k-\frac{3}{2}}2^{k-\frac{3}{2}}\Gamma(k-\frac{1}{2})}\frac{1}{2\pi i}\int_{(\sigma)}s^{\frac{1}{2}-k}e^{\frac{\pi\sqrt{DD'}}{2mc}(s-s^{-1})}ds
\end{aligned}$$

We have seen the remaining integral before and recall that it evaluates to $J_{k-\frac{3}{2}}(\frac{\pi}{mc}\sqrt{D'D})$. Inserting these revelations above gives

$$I_c = e^{-2\pi(nv+ry+n'v_0+r'y_0)}\frac{\pi^{k+\frac{1}{2}}}{(2m)^{k-\frac{1}{2}}\Gamma(k-\frac{1}{2})}\lambda_{k,m}(DD')^{\frac{k}{2}-\frac{3}{4}}H_{m,c}(n,-r,n',r')J_{k-\frac{3}{2}}(\frac{\pi}{mc}\sqrt{DD'}).$$

Collecting all the terms together (including the ones for $c<0$ which can be computed similarly) gives the desired result.