Contents

Finiteness of the set of solutions of Plateau's problem for polygonal boundary curves [1]

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Chapter 1

Introduction

Abstract
In this article it is proved that for a simple closed extreme polygon $\Gamma \subset \mathbb{R}^3$ every immersed stable minimal surface spanning Γ is an isolated point of the set of all minimal surfaces which span Γ, w. r. to the C^0-topology. Since the subset of immersed stable minimal surfaces spanning Γ is shown to be closed in the compact set of all minimal surfaces spanning Γ, this proves in particular that Γ can bound only <u>finitely</u> many immersed stable minimal surfaces. Moreover using this the author proves that for any extreme simple closed polygon $\Gamma^* \subset \mathbb{R}^3$, which meets the requirement that the angles at its vertices are different from $\frac{\pi}{2}$, there exists some neighborhood O of Γ^* in $\mathbb{R}^3$ and some integer β, depending on Γ^*, such that the number of immersed stable minimal surfaces spanning any simple closed polygon contained in O is bounded by β.

1.1 Main results

In 1978 Nitsche formulated the following conjecture (see [28]):
A "reasonably well behaved" simple closed contour can bound only <u>finitely</u> many solutions of Plateau's problem.
The first aim of the present article is a proof of the following partial result:

Theorem 1.1 *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed extreme polygon. Then every immersed stable minimal surface spanning Γ is an isolated point of the set of all minimal surfaces which span Γ, w. r. to the C^0-topology. In particular, Γ can bound only <u>finitely</u> many immersed stable minimal surfaces.*

We shall even extend this result by

Theorem 1.2 *Let $\Gamma^* \subset \mathbb{R}^3$ be an arbitrary extreme simple closed polygon, which meets the requirement that the angles at its vertices are different from*

$\frac{\pi}{2}$. Then there exists some neighborhood O of Γ^* in $\mathbb{R}^3$ and some integer β, depending on Γ^*, such that the number of immersed stable minimal surfaces spanning any simple closed polygon contained in O is bounded by β.

We term a polygon extreme if it is contained in the boundary of a compact convex subset of $\mathbb{R}^3$ and not entirely in a plane. Furthermore a disc-type minimal surface X is termed immersed if there holds

$$\inf_B \mid DX \mid > 0, \tag{1.1}$$

where we set $B := \{w = (u, v) \in \mathbb{R}^2 \mid\mid w \mid < 1\}$. It is additionally termed stable if the second variation of the area functional $\mathcal{A}$ in X in normal direction $\xi := \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$ satisfies

$$J^X(\varphi) := \int_B \mid \nabla\varphi \mid^2 + 2KE\,\varphi^2 dw = \frac{d^2}{d\epsilon^2}\mathcal{A}(X + \epsilon\varphi\,\xi) \mid_{\epsilon=0} \geq 0 \tag{1.2}$$

for any $\varphi \in C_c^\infty(B)$, where E denotes $\mid X_u \mid^2$ and $K \leq 0$ the Gaussian curvature of X.

1.2 Basic definitions and history

Let Γ be some closed piecewise linear Jordan curve in $\mathbb{R}^3$ being determined by the positions of its $N + 3$ vertices ($N \in \mathbb{N}$)

$$(P_1, ..., P_{N+3}) =: P \in \mathbb{R}^{3N+9}, \tag{1.3}$$

where we require the pairs of vectors $(P_{j+1} - P_j,\ P_j - P_{j-1})$ to be linear independent for $j = 1, \ldots, N + 3$, with $P_0 := P_{N+3}$ and $P_{N+4} := P_1$. We consider the "Plateau class" $\mathcal{C}^*(\Gamma)$ of surfaces $X \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$ that are spanned into Γ, i.e. whose boundary values $X \mid_{\partial B}: \mathbb{S}^1 \longrightarrow \Gamma$ are weakly monotonic mappings with degree equal to 1, satisfying a three-point-condition:

$$X(e^{i\tau_{N+k}}) = P_{N+k} \quad \text{for } \tau_{N+k} := \frac{\pi}{2}(1 + k),\ k = 1, 2, 3. \tag{1.4}$$

We endow $\mathcal{C}^*(\Gamma)$ with the norm $\| \cdot \|_{C^0(\bar{B})}$. Furthermore we denote by $(\mathcal{M}(\Gamma), \| \cdot \|_{C^0(\bar{B})})$ its subspace of all disc-type minimal surfaces, i.e. classical solutions X of the equations

$$\triangle X = 0 \tag{1.5}$$

$$\mid X_u \mid = \mid X_v \mid, \quad \langle X_u, X_v \rangle = 0 \tag{1.6}$$

on the disc B. In order to compare the results of this paper with already established theorems the reader shall translate the above definitions to simple closed boundary curves $\mathcal{R}$ in $\mathbb{R}^3$ of class $C^{4,\alpha}$, $\alpha \in (0, 1)$. The set of minimal surfaces spanning $\mathcal{R}$ which are additionally immersed and stable will be denoted by

$$\mathcal{M}_s(\mathcal{R}) := \{X \in \mathcal{M}(\mathcal{R}) \mid X \text{ satisfies } (1.1) \text{ and } (1.2)\}.$$

The first deep "finiteness result" was achieved by Tomi in [35]:

Theorem 1.3 (Tomi, 1973) *If $\mathcal{R}$ is a regular Jordan curve of class $C^{4,\alpha}$ with the property that all minimal surfaces $X \in \mathcal{M}(\mathcal{R})$, which yield global minimizers of the area functional $\mathcal{A}$ on $\mathcal{C}^*(\mathcal{R})$, are immersed, then there are only finitely many of them.*

Thus in combination with the papers [11] resp. [19] Tomi's theorem yields

Corollary 1.1 (Tomi, 1973) *If $\mathcal{R}$ is a regular Jordan curve of class $C^{4,\alpha}$ with total curvature less than 4π, or if $\mathcal{R}$ is analytic, then in $\mathcal{M}(\mathcal{R})$ there are only finitely many global minimizers of the area functional $\mathcal{A}$ on $\mathcal{C}^*(\mathcal{R})$.*

Four years later Tomi proved in [36]

Theorem 1.4 (Tomi, 1977) *If $\mathcal{R}$ is a proper curve, i.e. a regular Jordan curve of class $C^{4,\alpha}$ which bounds only minimal surfaces without boundary branch points and with interior branch points of at most first order (see Def. 2.1 below), then $\mathcal{M}_s(\mathcal{R})$ is finite. Moreover there exists some neighborhood O of $\mathcal{R}$ in $\mathbb{R}^3$ and some integer β, depending on $\mathcal{R}$, such that the number of immersed stable minimal surfaces spanning any proper curve contained in O is bounded by β.*

One year later Nitsche finally achieved his "6π-Theorem" in [28]:

Theorem 1.5 (Nitsche, 1978) *If $\mathcal{R}$ is a regular Jordan curve of class $C^{4,\alpha}$, which bounds only minimal surfaces without any branch points and whose total curvature does not exceed the value 6π, then the* entire *set $\mathcal{M}(\mathcal{R})$ is finite.*

Finally we shall pay attention to a result of Sauvigny in [31] which is most similar to the one of this article. There is only a difference with respect to the regularity of the boundary wire $\mathcal{R}$ and it is even stated for the set $\mathcal{H}^H_s(\mathcal{R})$ of immmersed, stable surfaces of constant mean curvature $H \in [0, 1)$, which however are assumed to be contained in the closed unit ball $\overline{B^3_1(0)}$, termed "small H-surfaces".

Theorem 1.6 (Sauvigny, 1989) *If $\mathcal{R}$ is an extreme, regular Jordan curve $\mathcal{R} \subset B^3_1(0)$ of class $C^{4,\alpha}$ and $H \in [0, 1)$, then it can bound only finitely many immersed "small" H-surfaces X, which are stable in the following generalized sense (compare with (1.2)):*

$$\int_B \mid \nabla\varphi \mid^2 \; + 2KE\,\varphi^2 - 4H^2E\,\varphi^2 dudv \geq 0 \quad \forall\, \varphi \in C^\infty_c(B).$$

It should be emphasized that the proofs of the above results depend on boundary regularity results for minimal surfaces resp. H-surfaces spanning $C^{4,\alpha}$-boundary curves due to Hildebrandt [20] resp. Heinz [12]. However the proof of the finiteness result of the present article is based on a combination of some of Tomi's and Sauvigny's ideas and results in [36], [31], [32] and [34] with Courant's and Heinz' achievements in [4], [14], [15], [16] and [17] on the representation of minimal surfaces that are bounded by a simple closed polygon and on their asymptotic behavior in the corners of the bounding polygon and with a deep connection between their total branch point orders, the defects of their assigned Schwarz operators and the number of vertices of the bounding polygon, expressed in Heinz' crucial formula (2.2) below, which is the main result of his outstanding paper [18]. We shall collect these tools in the following chapter.

Chapter 2

Courant's and Heinz' results

Our fundamental tools are Courant's [4] and Heinz' [14], [15] maps

$$\psi : T \longrightarrow (\mathcal{C}^*(\Gamma), \|\cdot\|_{C^0\cap H^{1,2}}),$$
$$\tilde{\psi} : T \longrightarrow C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3),$$

which are assigned to our arbitrarily fixed closed polygon Γ. Here T is an open bounded convex set of N-tupels

$$(\tau_1, \tau_2, \ldots, \tau_N) =: \tau \in (0, \pi)^N, \tag{2.1}$$

which meet the following chain of inequalities

$$0 < \tau_1 < \ldots < \tau_N < \pi = \tau_{N+1},$$

where $N+3$ was the number of vertices of the considered polygon (see (1.3)). Moreover to any $\tau \in T$ we assign the sets of surfaces

$$\mathcal{U}(\tau) := \{X \in \mathcal{C}^*(\Gamma) \mid X\mid_{\partial B}(e^{i\tau_j}) = P_j \quad \text{for } j = 1, \ldots, N\} \quad \text{and} \tag{2.2}$$
$$\tilde{\mathcal{U}}(\tau) := \{X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \mid X(e^{i\theta}) \in \Gamma_j \text{ for } \theta \in [\tau_j, \tau_{j+1}]\},$$

where we set $\Gamma_j := \{P_j + t\,(P_{j+1} - P_j) \mid t \in \mathbb{R}\}$, $P_{N+4} := P_1$ and $\tau_{N+4} := \tau_1$. On account of two uniqueness results in [4] resp. [14] one can define the maps

$$\psi(\tau) := \text{unique minimizer of } \mathcal{D} \text{ within } \mathcal{U}(\tau) \quad \text{and} \tag{2.3}$$
$$\tilde{\psi}(\tau) := \text{unique minimizer of } \mathcal{D} \text{ within } \tilde{\mathcal{U}}(\tau), \tag{2.4}$$

where $\mathcal{D}$ denotes Dirichlet's integral. We will also use the notation $X(\,\cdot\,, \tau)$ for $\tilde{\psi}(\tau)$. Now by the result of [4], (see also [23], p. 558) Satz 2 in [14] and the main theorem of [15], resp. Satz 1 in [17], these maps have the following properties:

Theorem 2.1 *(i) ψ is continuous on T.*
(ii) $f := \mathcal{D} \circ \psi$ is of class $C^1(T)$ and $\tilde{f} := \mathcal{D} \circ \tilde{\psi}$ even of class $C^\omega(T)$.
(iii) There holds $\tilde{f} \leq f$ on T and $\tilde{f}(\tau) = f(\tau)$ if and only if $\tilde{\psi}(\tau) = \psi(\tau)$, which is again equivalent to $\tilde{\psi}(\tau) \in \mathcal{C}^*(\Gamma)$.
(iv) $\tilde{\psi}(\tau)$ and $\psi(\tau)$ are harmonic on B $\forall\, \tau \in T$.
(v) The restriction

$$\psi\,|_{K(f)} \colon K(f) \xrightarrow{\cong} \mathcal{M}(\Gamma) \tag{2.5}$$

yields a homeomorphism *between the compact set of critical points of f and $(\mathcal{M}(\Gamma), \|\cdot\|_{C^0(\bar{B})})$ and a surface $\tilde{\psi}(\tau)$ is conformally parametrized on B, thus a minimal surface in $\tilde{\mathcal{U}}(\tau)$, if and only if $\tau \in K(\tilde{f})$.*
(vi) Let $\bar{\tau} \in T$ be arbitrarily fixed and $D \subset \mathbb{C}$ some simply connected domain whose intersection $D \cap B$ with B is nonvoid and connected and such that $\bar{D} \cap \{e^{i\bar{\tau}_k}\} = \emptyset$. Then there exists some neighborhood $U_D(\bar{\tau})$ of $\bar{\tau}$ in $\mathbb{C}^N$ and some holomorphic continuation of $X_w(\,\cdot\,,\,\cdot\,)$ onto $D \times U_D(\bar{\tau})$.
(vii) Furthermore for any $\bar{\tau} \in T$ and $k \in \{1, \ldots, N+3\}$ there exists some neighborhood $B_\delta(e^{i\bar{\tau}_k}) \times B_\delta^N(\bar{\tau})$ in $\mathbb{C} \times \mathbb{C}^N$ about $(e^{i\bar{\tau}_k}, \bar{\tau})$ such that there holds the representation:

$$X_w(w,\tau) = \sum_{j=1}^{p_k} f_j^k(w,\tau)\,(w - e^{i\tau_k})^{\rho_j^k} \tag{2.6}$$

for $(w,\tau) \in (B_\delta(e^{i\bar{\tau}_k}) \cap B) \times (B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N)$, where the the functions f_j^k are holomorphic on $B_\delta(e^{i\bar{\tau}_k}) \times B_\delta^N(\bar{\tau})$ and the exponents ρ_j^k satisfy

$$-1 < \rho_1^k < \ldots < \rho_{p_k}^k = 0, \qquad p_k \in \{2,3\}, \tag{2.7}$$

and do not depend on $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$.

The last assertion about the independence of the exponents ρ_j^k of $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$ follows immediately from [14], (2.20) and (3.28), as we point out now. We set $v_k := \frac{P_{k+1}-P_k}{|P_{k+1}-P_k|}$ and consider as in (2.20) of [14] the reflections S_k at the lines $\Gamma_k - P_k = Span(v_k)$ for $k \in \{1, \ldots, N+3\}$ (with $P_{N+4} := P_1$), explicitely given by

$$S_k(x) := -x + 2\,\langle v_k, x\rangle\, v_k \qquad \forall\, x \in \mathbb{R}^3.$$

The composed reflections $S_{k-1} \circ S_k \in \mathrm{SO}(3)$ are diagonalizable by conjugation with unitary matrizes and have eigenvalues on the $\mathbb{S}^1$. Now the ρ_j^k appear in (3.28) of [14] as pairwise different (negative) angles of these eigenvalues, precisely:

$$Spec(S_{k-1} \circ S_k) = \{e^{-2\pi i \rho_j^k}\}, \qquad 1 \leq j \leq p_k,$$

ordered as in (2.7) with $p_k \in \{2,3\}$, which proves the claimed independence of the exponents ρ_j^k of $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Moreover we shall note that

$S_{k-1} \circ S_k \neq \mathbf{1}$ and thus $p_k > 1$ by our requirement that the vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ have to be linearly independent. Moreover we see that $p_k = 2$ if and only if $\rho_1^k = -\frac{1}{2}$, i.e. if the spectrum of $S_{k-1} \circ S_k$ is $\{-1, -1, 1\}$, which can arise if and only if the angle β_k between the vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ is $\frac{\pi}{2}$. If in general $\beta_k \notin \{\frac{\pi}{2}, 0, \pi\}$, then the spectrum of $S_{k-1} \circ S_k$ is $\{\lambda, \bar{\lambda}, 1\}$ for some $\lambda \in \mathbb{S}^1$ with $\Im(\lambda) \neq 0$, i.e. $\rho_1^k + \rho_2^k = -1$. One can easily see that there holds either $-\rho_1^k \pi = \beta_k$ or $(\rho_1^k + 1)\pi = \beta_k$, which is by $\rho_1^k + \rho_2^k = -1$ equivalent to the pair of possibilities $(\rho_2^k + 1)\pi = \beta_k$ or $-\rho_2^k \pi = \beta_k$. Moreover we can expand the holomorphic functions f_j^k w. r. to w about the point $e^{i\tau_k}$ and obtain by (2.6) for any $k \in \{1, \ldots, N+3\}$:

$$X_w(w, \tau) = \sum_{j=1}^{p_k} \sum_{n=0}^{\infty} f_{j,n}^k(\tau)\,(w - e^{i\tau_k})^{\rho_j^k + n} \tag{2.8}$$

$\forall\, w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$ and $\forall\, \tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Now we fix some $\bar{\tau} \in T$ and choose that pair (j, n) for which $f_{j,n}^k(\bar{\tau}) \neq 0$ in (2.8) and $\rho_j^k + n$ is minimal and term this pair (j^*, m), i.e. we assign this pair to the point $\bar{\tau} \in T$. Since we know that either $(\rho_{j^*}^k + 1)\pi$ or $-\rho_{j^*}^k \pi$ equals the angle $\beta_k \neq 0, \pi$ between the linear independent vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ we conclude due to $\rho_{p_k}^k = 0$ that there has to hold $j^* < p_k$. Now using these terms we derive formula (2.5) in [18]:

Corollary 2.1 *For any fixed $\bar{\tau} \in T$ and $k \in \{1, \ldots, N+3\}$ there holds*

$$X_w(w, \bar{\tau}) = f_{j^*,m}^k(\bar{\tau})\,(w - e^{i\bar{\tau}_k})^{\rho_{j^*}^k + m} + O(\mid w - e^{i\bar{\tau}_k} \mid^{\rho_{j^*}^k + m + \epsilon}) \tag{2.9}$$

for $B \ni w \to e^{i\bar{\tau}_k}$, where $\epsilon := \rho_{j^*+1}^k - \rho_{j^*}^k \in (0, 1)$.

Proof: Firstly we note that ϵ is well defined by $j^* < p_k$. Now we split (2.8):

$$X_w(w, \bar{\tau}) = \sum_{j=1}^{p_k} f_{j,m}^k(\bar{\tau})\,(w - e^{i\bar{\tau}_k})^{\rho_j^k + m} + F^k(w, \bar{\tau}), \tag{2.10}$$

where we set

$$F^k(w, \bar{\tau}) := \sum_{j=1}^{p_k} \sum_{n=m+1}^{\infty} f_{j,n}^k(\bar{\tau})\,(w - e^{i\bar{\tau}_k})^{\rho_j^k + n}$$

and shall show that there holds:

$$\frac{\mid F^k(w, \bar{\tau}) \mid}{\mid w - e^{i\bar{\tau}_k} \mid^{\rho_{j^*}^k + m + \epsilon}} \longrightarrow 0 \qquad \text{for } w \to e^{i\bar{\tau}_k}. \tag{2.11}$$

Firstly we have by $\rho_{j^*}^k + \epsilon = \rho_{j^*+1}^k$:

$$\frac{\mid F^k(w, \bar{\tau}) \mid}{\mid w - e^{i\bar{\tau}_k} \mid^{\rho_{j^*}^k + m + \epsilon}} \leq \sum_{j=1}^{p_k} \sum_{n=1}^{\infty} \mid f_{j,n+m}^k(\bar{\tau}) \mid \mid w - e^{i\bar{\tau}_k} \mid^{n + \rho_j^k - \rho_{j^*+1}^k}$$

$\forall\, w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$. Now applying Cauchy's inequalities to the Taylor coefficients $f^k_{j,n+m}(\bar{\tau})$ we achieve:

$$\mid f^k_{j,n+m}(\bar{\tau}) \mid \leq \frac{c(\hat{\delta})}{\hat{\delta}^{n+m}} \qquad \forall\, n \in \mathbb{N},$$

for any $j \in \{1, \ldots, p_k\}$ and some arbitrarily fixed $\hat{\delta} \in (0, \delta)$ with

$$c(\hat{\delta}) := c(\hat{\delta}, \bar{\tau}, k) := \sqrt{3} \max_{j \in \{1,\ldots,p_k\}} \max_{\partial B_{\hat{\delta}}(e^{i\bar{\tau}_k})} \mid f^k_j(\,\cdot\,, \bar{\tau}) \mid .$$

Hence, using $p_k \leq 3$, $\rho^k_j \geq \rho^k_1$ and $\sum_{n=0}^{\infty} 2^{-n} = 2$ we obtain:

$$\frac{\mid F^k(w, \bar{\tau}) \mid}{\mid w - e^{i\bar{\tau}_k} \mid^{\rho^k_{j^*}+m+\epsilon}} \leq 3\, \frac{c(\hat{\delta})}{\hat{\delta}^{m+1}} \mid w - e^{i\bar{\tau}_k} \mid^{1+\rho^k_1 - \rho^k_{j^*+1}} \sum_{n=1}^{\infty} \frac{\mid w - e^{i\bar{\tau}_k} \mid^{n-1}}{\hat{\delta}^{n-1}}$$

$$\leq 3\, \frac{c(\hat{\delta})}{\hat{\delta}^{m+1}} \mid w - e^{i\bar{\tau}_k} \mid^{1+\rho^k_1 - \rho^k_{j^*+1}} \sum_{n=0}^{\infty} 2^{-n} = 6\, \frac{c(\hat{\delta})}{\hat{\delta}^{m+1}} \mid w - e^{i\bar{\tau}_k} \mid^{1+\rho^k_1 - \rho^k_{j^*+1}}$$

$\forall\, w \in B_{\frac{\hat{\delta}}{2}}(e^{i\bar{\tau}_k}) \cap B$. Thus taking $1 + \rho^k_1 - \rho^k_{j^*+1} > 0$ into account we achieve (2.11), which proves the corollary.

$\diamond$

Furthermore we derive from part (vi) of Theorem 2.1 that there is a Taylor expansion of $X_w(\,\cdot\,, \bar{\tau})$ about any point $w_0 \in \bar{B} \setminus \{e^{i\bar{\tau}_k}\}$:

$$X_w(w, \bar{\tau}) = a_m(\bar{\tau})\,(w - w_0)^m + a_{m+1}(\bar{\tau})\,(w - w_0)^{m+1} + \ldots \tag{2.12}$$

where the coefficients $\{a_j\}_{j \geq m}$ are holomorphic about the point $\bar{\tau}$ and $a_m(\bar{\tau}) \in \mathbb{C}^3 \setminus \{0\}$.

Definition 2.1 *(i) We term the exponent $m \equiv m^{\bar{\tau}}$ in (2.9) resp. (2.12) the branch point order of the surface $X(\,\cdot\,, \bar{\tau})$ at the point $e^{i\bar{\tau}_k}$, $k = 1, \ldots, N+3$, resp. $w_0 \in \bar{B} \setminus \{e^{i\bar{\tau}_k}\}$.*
(ii) A point $\bar{w} \in \bar{B}$ is termed a branch point of the minimal surface $X(\,\cdot\,, \bar{\tau}) \in \tilde{\mathcal{M}}(\Gamma)$ if its order $m^{\bar{\tau}}(\bar{w})$ is positive.

Hence, we see that there holds $m^{\bar{\tau}}(w) = 0$ in any point $w \in \bar{B}$ if and only if

$$\inf_B \mid DX(\,\cdot\,, \bar{\tau}) \mid > 0. \tag{2.13}$$

Furthermore one obtains easily by (2.9) and (2.12) that $X(\,\cdot\,, \bar{\tau})$ can have only finitely many branch points on $\bar{B}$. Hence, we may define its total branch point order

$$\kappa(\bar{\tau}) := \sum_{w \in B} m^{\bar{\tau}}(w) + \frac{1}{2} \sum_{w \in \partial B} m^{\bar{\tau}}(w).$$

Now we assign to every point $\tau \in K(\tilde{f})$, i.e. to every minimal surface $X(\,\cdot\,,\tau)$, its Schwarz operator

$$A^\tau \equiv A^{X(\,\cdot\,,\tau)} := -\triangle + 2\,(KE)^\tau,$$

where $(KE)^\tau(w) := (KE)(w,\tau)$ is defined as in (1.2), on the domain

$$Dom(A^\tau) := \{\varphi \in C^2(B) \cap \mathring{H}^{1,2}(B) \mid A^\tau(\varphi) \in L^2(B)\} \tag{2.14}$$

and formulate the central tool of the paper, "Heinz' formula", which is the main result of [18]:

Theorem 2.2 *For an arbitrary $\tau \in K(\tilde{f})$ there holds the formula*

$$\dim \operatorname{Ker} A^\tau + \operatorname{rank}(D^2\tilde{f}(\tau)) + 2\kappa(\tau) = N. \tag{2.15}$$

Next by differentiation of (2.8) w.r. to w we obtain formula (2.5') in [18] exactly as in the proof of Corollary 2.1, where one has to note that the components of $X_w(\,\cdot\,,\bar{\tau})$ are holomorphic functions due to $X_{w\bar{w}}(\,\cdot\,,\bar{\tau}) \equiv 0$ on B:

$$\begin{aligned} X_{ww}(w,\bar{\tau}) = f^k_{j^*,m}(\bar{\tau})\,(m+\rho^k_{j^*})\,(w-e^{i\bar{\tau}_k})^{\rho^k_{j^*}+m-1} \\ +O(\mid w-e^{i\bar{\tau}_k}\mid^{\rho^k_{j^*}+m+\epsilon-1}), \end{aligned} \tag{2.16}$$

for $B \ni w \to e^{i\bar{\tau}_k}$. Furthermore as stated in [18], formula (2.8), one can derive by integration of the holomorphic components of $X_w(\,\cdot\,,\bar{\tau})$ in (2.8) w. r. to w that for any fixed $\bar{\tau} \in T$ and $k \in \{1,\ldots,N+3\}$ there exists some $\delta > 0$ such that:

$$\begin{aligned} X(w,\tau) &= 2\,\Re\Big(\int_{e^{i\tau_k}}^{w} X_z(z,\tau)\,dz\Big) + X(e^{i\tau_k},\tau) \\ &= 2\,\Re\Big(\sum_{j=1}^{p_k}\sum_{l=0}^{\infty}\frac{f^k_{j,l}(\tau)}{\rho^k_j+l+1}\,(w-e^{i\tau_k})^{\rho^k_j+l+1}\Big) + P_k \\ &= \Re\Big(\sum_{j=1}^{p_k} g^k_j(w,\tau)\,(w-e^{i\tau_k})^{\rho^k_j+1}\Big) + P_k \end{aligned} \tag{2.17}$$

for $w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$ and $\tau \in B^N_\delta(\bar{\tau}) \cap \mathbb{R}^N$, where the functions

$$g^k_j(w,\tau) := \sum_{n=0}^{\infty}\frac{2\,f^k_{j,n}(\tau)}{\rho^k_j+n+1}\,(w-e^{i\tau_k})^n \tag{2.18}$$

are holomorphic on $B_\delta(e^{i\bar{\tau}_k}) \times B^N_\delta(\bar{\tau})$ and satisfy in particular $g^k_j(e^{i\tau_k},\tau) = \frac{2\,f^k_{j,0}(\tau)}{\rho^k_j+1}$. Next, as stated in formula (2.9) in [18], one achieves by differentiation of (2.17) w. r. to τ_l for $l \in \{1,\ldots,\hat{k},\ldots,N\}$:

$$X_{\tau_l}(w,\tau) = \Re\Big(\sum_{j=1}^{p_k}\frac{\partial g^k_j}{\partial \tau_l}(w,\tau)(w-e^{i\tau_k})^{\rho^k_j+1}\Big), \tag{2.19}$$

for $w \in B_\delta(e^{i\bar\tau_k}) \cap B$ and $\tau \in B^N_\delta(\bar\tau) \cap \mathbb{R}^N$ and for $l = k$:

$$X_{\tau_l}(w,\tau) = \Re\Big(\sum_{j=1}^{p_l} \frac{\partial g_j^l}{\partial \tau_l}(w,\tau)(w - e^{i\tau_l})^{\rho_j^l+1} - ie^{i\tau_l}\,(\rho_j^l+1)\, g_j^l(w,\tau)\,(w-e^{i\tau_l})^{\rho_j^l}\Big). \tag{2.20}$$

Moreover we shall verify now formula (2.10) in [18]. Let $\frac{\partial}{\partial\varphi} := u\,\frac{\partial}{\partial v} - v\,\frac{\partial}{\partial u}$ denote differentiation w. r. to the angle φ. Firstly we compute for some $l = k \in \{1,\dots,N\}$ and $j \in \{1,\dots,p_l\}$:

$$\begin{aligned}\frac{\partial}{\partial\varphi}(w - e^{i\tau_l})^{\rho_j^l+1} &= u\,(\rho_j^l+1)\,(w-e^{i\tau_l})^{\rho_j^l}\, i - v\,(\rho_j^l+1)\,(w-e^{i\tau_l})^{\rho_j^l}\, 1 \\ &= (\rho_j^l+1)\,(w-e^{i\tau_l})^{\rho_j^l}\, iw.\end{aligned} \tag{2.21}$$

Thus together with (2.20) we achieve as in (2.10) in [18] for the case $l = k$:

$$\begin{aligned} & X_{\tau_l}(w,\tau) + X_\varphi(w,\tau) \\ &= \Re\Big(\sum_{j=1}^{p_l} \Big(\frac{\partial g_j^l}{\partial\tau_l}(w,\tau) + \frac{\partial g_j^l}{\partial\varphi}(w,\tau)\Big)(w - e^{i\tau_l})^{\rho_j^l+1} \\ &\qquad + (-ie^{i\tau_l} + iw)\,(\rho_j^l+1)\, g_j^l(w,\tau)\,(w-e^{i\tau_l})^{\rho_j^l}\Big) \\ &= \Re\Big(\sum_{j=1}^{p_l} \Big(\frac{\partial g_j^l}{\partial\tau_l}(w,\tau) + \frac{\partial g_j^l}{\partial\varphi}(w,\tau) + i\,(\rho_j^l+1)\, g_j^l(w,\tau)\Big)(w-e^{i\tau_l})^{\rho_j^l+1}\Big)\end{aligned} \tag{2.22}$$

for $w \in B_\delta(e^{i\bar\tau_l}) \cap B$ and $\tau \in B^N_\delta(\bar\tau) \cap \mathbb{R}^N$. Combining this with (2.19) we achieve

Corollary 2.2 *Let $\bar\tau \in K(\tilde f)$ be arbitrarily chosen. If there holds $m^{\bar\tau}(e^{i\bar\tau_l}) = 0$ for each $l \in \{1,\dots,N\}$, then the functions $X_{\tau_l}(\,\cdot\,,\bar\tau)$ are linearly independent on B.*

Proof: We assume that the assertion was wrong, i.e. that there exists some linear relation

$$\sum_{l=1}^N \alpha_l\, X_{\tau_l}(\,\cdot\,,\bar\tau) \equiv 0 \qquad \textit{on } B, \tag{2.23}$$

where there is at least one index $k \in \{1,\dots,N\}$ with $\alpha_k \neq 0$. By (2.19) we see that $X_{\tau_l}(w,\bar\tau) \longrightarrow 0$ for $w \to e^{i\bar\tau_k}$ and $l \neq k$. Hence, inserting this into (2.23) we obtain due to $\alpha_k \neq 0$:

$$X_{\tau_k}(w,\bar\tau) \longrightarrow 0 \qquad \textit{for } w \to e^{i\bar\tau_k}.$$

Now together with (2.22) we conclude that there holds also:

$$X_\varphi(w,\bar\tau) \longrightarrow 0 \qquad \textit{for } w \to e^{i\bar\tau_k}. \tag{2.24}$$

Moreover as we require $\bar{\tau} \in K(\tilde{f})$, thus that $X(\,\cdot\,,\bar{\tau})$ is conformally parametrized on B, we have by $\frac{\partial}{\partial\varphi} := u\,\frac{\partial}{\partial v} - v\,\frac{\partial}{\partial u}$:

$$\begin{aligned} \mid X_{\varphi}(w,\bar{\tau}) \mid^2 = u^2 \mid X_v(w,\bar{\tau}) \mid^2 - 2\,u\,v\,\langle X_v(w,\bar{\tau}), X_u(w,\bar{\tau})\rangle + v^2 \mid X_u(w,\bar{\tau}) \mid^2 \\ = 2 \mid w \mid^2 \mid X_w(w,\bar{\tau}) \mid^2 \qquad \forall\, w \in B. \end{aligned}$$

Hence, in combination with (2.24) we obtain finally:

$$X_w(w,\bar{\tau}) \longrightarrow 0 \qquad \textit{for } w \to e^{i\bar{\tau}_k},$$

which implies $m^{\bar{\tau}}(e^{i\bar{\tau}_k}) > 0$ by (2.9) contradicting the requirement of the corollary and proves our assertion.

$\diamond$

Now we set

$$\rho := \min_{k=1,\dots,N+3} \rho_1^k > -1. \tag{2.25}$$

By the Courant-Lebesgue Lemma and point (iv) of Theorem 2.1 we shall prove the following important

Lemma 2.1 *There holds*

$$\begin{aligned} \tilde{\mathcal{M}}(\Gamma) := \{\text{set of minimal surfaces on } B\} \cap \bigcup_{\tau\in T} \tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B,\mathbb{R}^3) \\ = \{X \in \text{image}(\tilde{\psi}) \mid X \text{ satisfies also } (1.6)\}. \end{aligned}$$

Proof: We fix some $\tau \in T$ arbitrarily. By the definition of $\tilde{\psi}$, i.e. by $\mathcal{D}(\tilde{\psi}(\tau)) = \inf_{\tilde{\mathcal{U}}(\tau)} \mathcal{D} < \infty$, and $\tilde{\psi}(\tau) \in C^2(B,\mathbb{R}^3)$ we see that $\tilde{\psi}(\tau) \in H^{1,2}(B,\mathbb{R}^3)$. Thus together with point (iv) of Theorem 2.1 we can derive the inclusion $''\supset''$ of our assertion and secondly by the definition of $\tilde{\psi}(\tau)$ that

$$\mathcal{D}(\tilde{\psi}(\tau)) = \inf_{\tilde{\mathcal{U}}(\tau)} \mathcal{D} \leq \inf_{\tilde{\mathcal{U}}(\tau)\cap H^{1,2}(B,\mathbb{R}^3)} \mathcal{D} \leq \mathcal{D}(\tilde{\psi}(\tau))$$

thus

$$\inf_{\tilde{\mathcal{U}}(\tau)} \mathcal{D} = \inf_{\tilde{\mathcal{U}}(\tau)\cap H^{1,2}(B,\mathbb{R}^3)} \mathcal{D}. \tag{2.26}$$

Hence, in order to prove the inclusion $''\subset''$ it suffices to prove that some arbitrary minimal surface $X \in \tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B,\mathbb{R}^3)$ satisfies

$$\mathcal{D}(X) < \mathcal{D}(Y) \qquad \forall\, Y \in (\tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B,\mathbb{R}^3)) \setminus \{X\}, \tag{2.27}$$

since $\tilde{\psi}(\tau)$ is the unique minimizer of $\mathcal{D}$ in $\tilde{\mathcal{U}}(\tau)$ and therefore $X = \tilde{\psi}(\tau)$. Now we choose some arbitrary $Y \in (\tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B,\mathbb{R}^3)) \setminus \{X\}$ and set $Z := Y - X$. We choose for each $k \in \{1,\dots,N+3\}$ some radius $r^k \in (0,\frac{1}{4})$ and consider the domain $\Omega(\tau) := B \setminus \bigcup_{k=1}^{N+3} \overline{B_{r^k}(e^{i\tau_k})}$. Since there holds $X \in$

$C^\infty(\overline{\Omega(\tau)}, \mathbb{R}^3)$ and $Z \in H^{1,2}(B, \mathbb{R}^3)$ we may apply the divergence theorem to $Z \cdot DX$ on $\Omega(\tau)$, which yields:

$$\int_{\partial\Omega(\tau)} \langle Z, \frac{\partial}{\partial\nu} X\rangle \, ds = \int_{\Omega(\tau)} \langle Z, \triangle X\rangle \, dw + \int_{\Omega(\tau)} DZ \cdot DX \, dw, \tag{2.28}$$

where ν denotes the outward unit normal about $\partial\Omega(\tau)$. On account of

$$\mathcal{D}_{\Omega(\tau)}(Y) = \mathcal{D}_{\Omega(\tau)}(Z + X) = \mathcal{D}_{\Omega(\tau)}(Z) + \int_{\Omega(\tau)} DZ \cdot DX \, dw + \mathcal{D}_{\Omega(\tau)}(X)$$

and $\triangle X \equiv 0$ on B we achieve therefore:

$$\mathcal{D}_{\Omega(\tau)}(X) = \mathcal{D}_{\Omega(\tau)}(Y) - \mathcal{D}_{\Omega(\tau)}(Z) - \int_{\partial\Omega(\tau)} \langle Z, \frac{\partial}{\partial\nu} X\rangle \, ds. \tag{2.29}$$

Now we choose some $l \in \{1, \dots, N+3\}$ arbitrarily and consider the adjacent pair (τ_l, τ_{l+1}) and a rotation $S \in SO(3)$ which satisfies $S(P_{l+1} - P_l) = (0, 0, \mid P_{l+1} - P_l \mid)$, i.e. which turns the line $\Gamma_l - P_l$ onto the x_3-axis. Hence, terming $\bar{X} := S(X)$ and $\bar{Z} := S(Z)$ we achieve by $X, Y \in \tilde{\mathcal{U}}(\tau)$ and $Z = X - Y$:

$$\bar{Z}^1(e^{i\varphi}) \equiv 0 \equiv \bar{Z}^2(e^{i\varphi}) \qquad \forall\, \varphi \in [\tau_l, \tau_{l+1}]. \tag{2.30}$$

Moreover we conclude from $X(e^{i\varphi}) \in \Gamma_l$, $\forall\, \varphi \in [\tau_l, \tau_{l+1}]$, that $X_\varphi(e^{i\varphi}) \in \Gamma_l - P_l$, $\forall\, \varphi \in (\tau_l, \tau_{l+1})$, which yields by the conformality of X, especially by $\langle X_\varphi, X_r\rangle \equiv 0$ on $\bar{B} \setminus \{e^{i\tau_k}\}_{k=1,\dots,N+3}$, that

$$X_r(e^{i\varphi}) \perp \Gamma_l - P_l \qquad \forall\, \varphi \in (\tau_l, \tau_{l+1}). \tag{2.31}$$

From this we achieve immediately that $\bar{X}_r^3(e^{i\varphi}) \equiv 0 \;\; \forall\, \varphi \in (\tau_l, \tau_{l+1})$. Hence, together with (2.30) we arrive at

$$\langle Z, X_r\rangle(e^{i\varphi}) = \langle \bar{Z}, \bar{X}_r\rangle(e^{i\varphi}) \equiv 0 \qquad \forall\, \varphi \in (\tau_l, \tau_{l+1}),$$

for any fixed $l \in \{1, \dots, N+3\}$, and therefore

$$\int_{\partial\Omega(\tau)} \langle Z, \frac{\partial}{\partial\nu} X\rangle \, ds = \sum_{k=1}^{N+3} \int_{C_{r_k}(e^{i\tau_k})} \langle Z, \frac{\partial}{\partial\nu} X\rangle \, ds, \tag{2.32}$$

where we denote

$$C_r(e^{i\tau_l}) := \bar{B} \cap \partial B_r(e^{i\tau_l}) = \{e^{i\tau_l} + r\, e^{i\theta} \mid \theta_1(r) \le \theta \le \theta_2(r)\},$$

for $r \in (0, \frac{1}{4})$, and introduce polar coordinates about the point $e^{i\tau_l}$, i.e. $\tilde{X}(r, \theta) := X(e^{i\tau_l} + r\, e^{i\theta})$ and $\tilde{Z}(r, \theta) := Z(e^{i\tau_l} + r\, e^{i\theta})$, for $\theta \in [\theta_1(r), \theta_2(r)]$ and any $r \in (0, \frac{1}{4})$. Now due to $\frac{\partial}{\partial\nu} X(e^{i\tau_l} + r\, e^{i\theta}) = -\tilde{X}_r(r, \theta)$ for any $\theta \in [\theta_1(r), \theta_2(r)]$ we are lead to examine the behaviour of the integrals

$$I(r) \equiv I^l(r) := \int_{\theta_1(r)}^{\theta_2(r)} \mid \tilde{Z}(r, \theta) \mid \mid \tilde{X}_r(r, \theta) \mid \; r \, d\theta,$$

for $r \searrow 0$. We compute

$$\tilde{X}_r(r,\theta) = X_u(e^{i\tau_l} + r\,e^{i\theta})\,\cos\theta + X_v(e^{i\tau_l} + r\,e^{i\theta})\,\sin\theta,$$
$$\tilde{X}_\theta(r,\theta) = r\,(-X_u(e^{i\tau_l} + r\,e^{i\theta})\,\sin\theta + X_v(e^{i\tau_l} + r\,e^{i\theta})\,\cos\theta),$$

which yields by the conformality of X:

$$\mid \tilde{X}_\theta(r,\theta) \mid^2 = r^2 \mid X_u(e^{i\tau_l} + r\,e^{i\theta}) \mid^2 = r^2 \mid \tilde{X}_r(r,\theta) \mid^2,$$

for any $\theta \in [\theta_1(r), \theta_2(r)]$ and $r \in (0, \frac{1}{4})$. Together with the Courant-Lebesgue Lemma we achieve for any $\delta \in (0, \frac{1}{4})$ the existence of some $\xi \in (\delta, \sqrt{\delta})$ such that

$$I(\xi) \leq \parallel Z \parallel_{C^0(\bar{B})} \int_{\theta_1(\xi)}^{\theta_2(\xi)} \mid \tilde{X}_r(\xi,\theta) \mid \,\xi\, d\theta$$

$$= \parallel Z \parallel_{C^0(\bar{B})} \int_{\theta_1(\xi)}^{\theta_2(\xi)} \mid \tilde{X}_\theta(\xi,\theta) \mid \,d\theta \leq \parallel Z \parallel_{C^0(\bar{B})} \sqrt{\frac{4\,\mathcal{D}(X)}{\log(\frac{1}{\delta})}\,\mid \theta_1(\xi) - \theta_2(\xi) \mid}.$$

Thus choosing some sequence $\delta_n \searrow 0$ we obtain a null sequence of radii $\xi_n^l \in (\delta_n, \sqrt{\delta_n})$ such that

$$\int_{\theta_1(\xi_n^l)}^{\theta_2(\xi_n^l)} \mid \tilde{Z}(\xi_n^l,\theta) \mid \mid \tilde{X}_r(\xi_n^l,\theta) \mid \,\xi_n^l\, d\theta = I^l(\xi_n^l) \longrightarrow 0 \qquad \textit{for } n \to \infty, \tag{2.33}$$

for any fixed $l \in \{1, \ldots, N+3\}$. Thus choosing for each l such an appropriate null sequence $\{\xi_n^l\}$ which satisfies (2.33) and setting $\Omega^n(\tau) := B \setminus \bigcup_{k=1}^{N+3} \overline{B_{\xi_n^k}(e^{i\tau_k})}$ we obtain by (2.32):

$$\mid \int_{\partial\Omega^n(\tau)} \langle Z, \frac{\partial}{\partial\nu} X \rangle\, ds \mid \leq \sum_{k=1}^{N+3} \int_{C_{\xi_n^k}(e^{i\tau_k})} \mid \langle Z, \frac{\partial}{\partial\nu} X \rangle \mid \, ds \longrightarrow 0$$

for $n \to \infty$. Hence, inserting this into (2.29) and letting tend $n \to \infty$ we achieve in fact (2.27) due to $X, Y, Z \in H^{1,2}(B, \mathbb{R}^3)$:

$$\mathcal{D}(X) = \mathcal{D}(Y) - \mathcal{D}(Z) < \mathcal{D}(Y),$$

since $\mathcal{D}(Z) = 0$ would imply that $Z \equiv$ *const.* on $\bar{B}$ with *const.* $= 0$ by $Z(e^{i\tau_l}) = (X - Y)(e^{i\tau_l}) = P_l - P_l = 0$ for each l, in contradiction to our requirement that $Y \not\equiv X$ on $\bar{B}$.

$\diamond$

Combining this result with (2.12) we achieve

Corollary 2.3 *Let $\bar{\tau} \in T$ be arbitrarily fixed. Then the intersection $\tilde{\mathcal{M}}(\Gamma) \cap \tilde{\mathcal{U}}(\bar{\tau})$ is either empty or consists of the single point $\tilde{\psi}(\bar{\tau})$.*

Proof: We suppose $\tilde{\mathcal{M}}(\Gamma) \cap \tilde{\mathcal{U}}(\bar{\tau})$ to be nonvoid and the assertion of the corollary to be wrong. Thus $\tilde{\mathcal{M}}(\Gamma) \cap \tilde{\mathcal{U}}(\bar{\tau})$ would have to contain an element X which is different from $\tilde{\psi}(\bar{\tau})$. Now by Lemma 2.1 this assertion implies the existence of some point $\hat{\tau} \in T \setminus \{\bar{\tau}\}$ whose image $\tilde{\psi}(\hat{\tau})$ yields this surface X, thus a minimal surface lying in the intersection $\tilde{\mathcal{U}}(\bar{\tau}) \cap \tilde{\mathcal{U}}(\hat{\tau})$. By $\bar{\tau} \neq \hat{\tau}$ there has to be an index $j \in \{1, \ldots, N\}$ such that $\bar{\tau}_j \neq \hat{\tau}_j$, say $\bar{\tau}_j < \hat{\tau}_j$ without loss of generality. Now by the definition of the sets $\tilde{\mathcal{U}}(\tau)$ and the linear independence of any two adjacent edges of Γ one can easily deduce that $X \equiv \tilde{\psi}(\hat{\tau})$ would have to map the entire arc $\gamma := \{e^{i\theta} \mid \bar{\tau}_j \leq \theta \leq \hat{\tau}_j\}$ into the vertex P_j of Γ. Moreover since point (vi) of Theorem 2.1 guarantees that the derivatives $D_{(u,v)}X$ can be continued continuously onto $\bar{B} \setminus \{e^{i\hat{\tau}_k}\}_{k=1,\ldots,N+3}$ the conformality relations of X still hold especially on some open subarc $\hat{\gamma} \subset \gamma$. Thus we obtain that $\frac{d}{d\theta}X(e^{i\theta}) \equiv 0$ for $e^{i\theta} \in \hat{\gamma}$ and therefore:

$$
\begin{aligned}
0 \equiv \left| \frac{d}{d\theta} X((\cos\theta, \sin\theta)) \right|^2 &= \left| -\sin(\theta)\, X_u(e^{i\theta}) + \cos(\theta)\, X_v(e^{i\theta}) \right|^2 \\
&= \sin(\theta)^2 \; | X_u(e^{i\theta}) |^2 - 2 \sin(\theta) \cos(\theta) \, \langle X_u(e^{i\theta}), X_v(e^{i\theta}) \rangle \\
&\qquad + \cos(\theta)^2 \; | X_v(e^{i\theta}) |^2 = | X_u(e^{i\theta}) |^2 = | X_v(e^{i\theta}) |^2,
\end{aligned}
$$

i.e. $D_{(u,v)}X \equiv 0$ on $\hat{\gamma}$, showing that $\hat{\gamma}$ consists of boundary branch points of $X \equiv \tilde{\psi}(\hat{\tau})$. But this contradicts the power series expansion (2.12) of X about an arbitrarily chosen point on $\hat{\gamma}$, forcing such a branch point to be isolated. This proves the corollary.

$\diamond$

Together with point (v) of Theorem 2.1 we can derive the following crucial

Corollary 2.4 *There holds* $\mathcal{M}(\Gamma) \subset \tilde{\mathcal{M}}(\Gamma)$ *and also* $K(f) \subset K(\tilde{f})$. *In particular* $X(\,\cdot\,, \tau) \equiv \tilde{\psi}(\tau)$ *coincides with* $\psi(\tau)$ *for any* $\tau \in K(f)$.

Proof: Firstly the inclusion $\mathcal{M}(\Gamma) \subset \tilde{\mathcal{M}}(\Gamma)$ is trivial. Now let $\bar{\tau} \in K(f)$ be arbitrarily chosen. Then we obtain $\psi(\bar{\tau}) \in \mathcal{M}(\Gamma)$ by (2.5) and therefore also $\psi(\bar{\tau}) \in \tilde{\mathcal{M}}(\Gamma)$. As we also know that $\psi(\bar{\tau}) \in \tilde{\mathcal{U}}(\bar{\tau})$ by the definition of $\tilde{\mathcal{U}}(\bar{\tau})$ we conclude that $\tilde{\mathcal{M}}(\Gamma) \cap \tilde{\mathcal{U}}(\bar{\tau})$ contains $\psi(\bar{\tau})$ and is nonvoid in particular. By the above corollary this proves $\psi(\bar{\tau}) = \tilde{\psi}(\bar{\tau})$ and hence already the last statement of the corollary. Finally the conformality of $\tilde{\psi}(\bar{\tau})$ on B, that we have just proved, is equivalent to $\bar{\tau} \in \tilde{K}(f)$ by point (v) of Theorem 2.1, which shows indeed $K(f) \subset K(\tilde{f})$.

$\diamond$

Moreover let

$$
\xi(\,\cdot\,, \tau) := \frac{X_u \wedge X_v}{| X_u \wedge X_v |}(\,\cdot\,, \tau) = \frac{X_w \wedge X_{\bar{w}}}{i \, | X_w |^2}(\,\cdot\,, \tau) \tag{2.34}
$$

denote the unit normal field of some minimal surface $X(\,\cdot\,,\tau) \in \tilde{\mathcal{M}}(\Gamma)$, i.e. for some $\tau \in K(\tilde{f})$. By (2.9) and (2.12) one achieves that $\xi(\,\cdot\,,\tau)$ can be continued continuously onto $\bar{B}$ and even analytically onto $\bar{B} \setminus \{e^{i\tau_l}\}_{l=1,\dots,N+3}$, although it is not defined in the branch points of $X(\,\cdot\,,\tau)$, and that at some point $e^{i\tau_k}$ it behaves asymptotically like

$$\xi(w,\tau) = \frac{f^k_{j^*,m}(\tau) \wedge \overline{f^k_{j^*,m}(\tau)}}{i \mid f^k_{j^*,m}(\tau) \mid^2} + O(\mid w - e^{i\tau_k} \mid^{\epsilon}) \qquad \textit{for } w \longrightarrow e^{i\tau_k} \tag{2.35}$$

and $k = 1, \dots, N+3$. Together with (2.16) one obtains moreover:

$$\frac{\mid \langle \xi(w,\tau), X_{ww}(w,\tau) \rangle \mid}{\mid X_w(w,\tau) \mid} = O(\mid w - e^{i\tau_k} \mid^{\epsilon - 1}) \qquad \textit{for } w \longrightarrow e^{i\tau_k} \tag{2.36}$$

and $k = 1, \dots, N+3$. We can use this and identity (3.2) in [18]

$$(KE)^\tau(w) \equiv (KE)(\tau, w) = -\frac{8 \mid \langle \xi(w,\tau), X_{ww}(w,\tau) \rangle \mid^2}{\mid X_w(w,\tau) \mid^2} \tag{2.37}$$

for any $\tau \in K(\tilde{f})$, for the product of the Gauss curvature and the surface element of $X(\,\cdot\,,\tau)$ as in (1.2), in order to derive estimate (3.3) in [18], i.e. that there is some constant *const.*(τ), depending on τ and Γ only, such that:

$$\mid (KE)^\tau(w) \mid \leq \textit{const.}(\tau) \sum_{k=1}^{N+3} \mid w - e^{i\tau_k} \mid^{-2+\alpha} \qquad \forall\, w \in B, \tag{2.38}$$

for

$$\alpha := 2 \min\{\rho^k_j - \rho^k_{j-1} \mid j = 1, \dots, p_k, k = 1, \dots, N+3\} > 0 \tag{2.39}$$

where we set $\rho^k_0 := -1$ for each k. Thus we note that α does not depend on $\tau \in K(\tilde{f})$, as the ρ^k_j do not. This implies especially that $(KE)^\tau \in L^{p^*}(B)$, for any $p^* \in (1, \frac{2}{2-\alpha})$ and any $\tau \in K(\tilde{f})$. Moreover one can insure by (2.12) and (2.37) that the function $(KE)^\tau$, which is not defined in the branch points of $X(\,\cdot\,,\tau)$, is of class $L^\infty_{loc}(\bar{B} \setminus \{e^{i\tau_l}\}_{l=1,\dots,N+3})$ and can in fact be continued analytically onto $\bar{B} \setminus \{e^{i\tau_l}\}_{l=1,\dots,N+3}$.

Chapter 3

Compactness of $\mathcal{M}_S(\Gamma)$

In this chapter we prove

Theorem 3.1 *If Γ is a rectifiable closed Jordan curve in $\mathbb{R}^3$ which bounds only minimal surfaces without boundary branch points, then $\mathcal{M}_s(\Gamma)$ is a closed subset of $\mathcal{M}(\Gamma)$, thus compact, w. r. to the $C^0(\bar{B})$-topology.*

The proof consists of two parts. The first is

Proposition 3.1 *Let $\{X^n\}$ be a sequence of stable minimal surfaces defined on B without interior branch points and with*

$$X^n \longrightarrow X \qquad \text{in } C^1_{loc}(B, \mathbb{R}^3), \tag{3.1}$$

for some minimal surface X on B. Then X is stable again.

Proof: We fix some $\varphi \in C_c^\infty(B)$ arbitrarily. On account of the requirement $\mid X_w^n \mid > 0$ on B we can use the identity $(KE)^n = -\frac{8\,|\langle \xi^n, X^n_{ww}\rangle|^2}{|X^n_w|^2}$ in order to conclude by (3.1) and Cauchy's estimates that

$$(KE)^n\,\varphi^2(w) \longrightarrow KE\,\varphi^2(w) \qquad \text{pointwise for a.e.}\, w \in B$$

and for $n \to \infty$. Now together with $-(KE)^n \geq 0$ on B and $J^{X^n}(\varphi) \geq 0$ for any $n \in \mathbb{N}$ we achieve by Fatou's lemma:

$$\int_B -2\,KE\,\varphi^2\,dw \leq \liminf_{n\to\infty} \int_B -2\,(KE)^n \varphi^2\,dw \leq \int_B \mid \nabla\varphi \mid^2\ dw,$$

i.e. $J^X(\varphi) \geq 0$ for any $\varphi \in C_c^\infty(B)$, which proves the stability of X.

$\diamond$

Furthermore we derive from Theorem 1 in [31] for $H = 0$:

Proposition 3.2 *Let $\{X^n\}$ be some sequence of stable minimal surfaces defined on B without interior branch points and with*

$$X^n \longrightarrow X \qquad \text{in } C^1_{loc}(B, \mathbb{R}^3),$$

for some non-constant minimal surface X. Then X does not possess any interior branch points neither.

Now Theorem 3.1 is an immediate consequence of Prop. 3.1 and 3.2 taking into account that $\mathcal{M}(\Gamma)$ in fact does not contain any constant minimal surfaces due to the three-point condition and is compact w. r. to the $C^0(\bar{B})$-topology and that Γ is required to bound only minimal surfaces without boundary branch points.

Finally we shall prove the following easy approximation result to be used below in Chapters 10 and 11:

Lemma 3.1 *If X is some stable minimal surface, i.e. if there holds $J^X(\varphi) \geq 0 \;\; \forall\, \varphi \in C^\infty_c(B)$, then this inequality extends to all functions $\varphi \in \mathring{H}^{1,2}(B)$.*

Proof: Let some $\varphi \in \mathring{H}^{1,2}(B)$ be chosen arbitrarily and $\{\varphi_j\} \subset C^\infty_c(B)$ some sequence with

$$\varphi_j \longrightarrow \varphi \qquad \textit{in } \mathring{H}^{1,2}(B). \tag{3.2}$$

By Sobolev's embedding theorem we have $H^{1,2}(B) \hookrightarrow L^q(B),\ \forall\, q \in [1, \infty)$, due to $1 - \frac{2}{2} = 0 > 0 - \frac{2}{q}$, which implies $\varphi_j \longrightarrow \varphi$ in $L^q(B)$, for any $q \in [1, \infty)$, and therefore together with Hölder's inequality:

$$\| KE\,(\varphi_j^2 - \varphi^2) \|_{L^1(B)} \leq \| KE \|_{L^{p^*}(B)} \| \varphi_j^2 - \varphi^2 \|_{L^{p'}(B)} \longrightarrow 0, \tag{3.3}$$

for $j \to \infty$, with $\frac{1}{p^*} + \frac{1}{p'} = 1$. Thus together with the required stability of X and (3.2) we obtain:

$$0 \leq J^X(\varphi_j) \longrightarrow J^X(\varphi) \geq 0 \qquad \textit{for } j \to \infty.$$

$\diamond$

Chapter 4

Extreme polygons prevent boundary branch points

In this chapter we shall show that Theorem 3.1 especially applies to *extreme* polygons. Firstly we define as in [28], p. 146:

Definition 4.1 *We term a polygon Γ extreme if it lies on the boundary of some compact convex set $K \subset \mathbb{R}^3$ and is not contained in a plane.*

Remark 4.1 *One verifies easily that the first requirement in the above definition is equivalent to the so-called "boundary k-point Radó condition" for $k = 0$, which just asserts that each point of Γ possesses a supporting plane w. r. to Γ, i.e. a plane determining a closed half space which contains Γ entirely.*

Firstly we need the following generalization of Hopf's "boundary point lemma":

Lemma 4.1 *Let $\Phi \in C^0(\bar{B}) \cap C^2(B)$ be harmonic on B and satisfy*

$$\Phi(w_0) > \Phi(w) \qquad \forall\, w \in B, \tag{4.1}$$

for some fixed point $w_0 \in \partial B$. Then there exists some constant $\sigma > 0$ such that there holds

$$\frac{\Phi(w_0) - \Phi(w)}{\mid w_0 - w \mid} > \sigma \qquad \forall\, w \in K_{\delta,\frac{\pi}{4}}(w_0), \tag{4.2}$$

for some sufficiently small chosen $\delta > 0$, where $K_{\delta,\frac{\pi}{4}}(w_0) := \{w \in B_\delta(w_0) \cap B \mid\mid \text{angle}(w - w_0, -w_0) \mid\in [0, \frac{\pi}{4}]\}$.

Proof: We consider the harmonic function $\tilde{\Phi} := \Phi - \Phi(w_0)$, satisfying $\tilde{\Phi} < 0$ on B by (4.1). We choose some disc $B_R(w^*) \subset B$ with $\overline{B_R(w^*)} \cap \partial B = \{w_0\}$ and consider for some arbitrary $\alpha > 0$ the function

$$v(w) := e^{-\alpha|w-w^*|^2} - e^{-\alpha R^2} \qquad \textit{for } w \in \overline{B_R(w^*)}.$$

Setting $\tilde{v}(r) := v(w)$ for $r :=\mid w - w^* \mid\leq R$ one easily computes:

$$\triangle v = \frac{1}{r}\tilde{v}_r + \tilde{v}_{rr} = 4\alpha\,(\alpha\, r^2 - 1)\, e^{-\alpha r^2} \qquad on\ B_R(w^*).$$

Hence, choosing some $\alpha \geq \frac{4}{R^2}$ we obtain for any $\epsilon > 0$:

$$\triangle(\tilde{\Phi} + \epsilon v) = \epsilon\,\triangle v > \epsilon\, 4\alpha\left(\alpha\,\frac{R^2}{4} - 1\right) e^{-\alpha R^2} \geq 0 \qquad on\ T_R(w^*), \tag{4.3}$$

with $T_R(w^*) := B_R(w^*)\setminus\overline{B_{\frac{R}{2}}(w^*)}$. Now by $\max_{\partial B_{\frac{R}{2}}(w^*)}\tilde{\Phi} < 0$ we can choose $\epsilon > 0$ sufficiently small such that still $\tilde{\Phi} + \epsilon v \leq 0$ on $\partial B_{\frac{R}{2}}(w^*)$ and together with $v \equiv 0$ and $\tilde{\Phi} \leq 0$ on $\partial B_R(w^*)$ we can conclude that $\tilde{\Phi} + \epsilon v \leq 0$ on $\partial T_R(w^*)$. Now in combination with (4.3) the weak maximum principle for subharmonic functions yields that

$$\tilde{\Phi} + \epsilon v \leq 0 \qquad on\ T_R(w^*), \tag{4.4}$$

for $\epsilon > 0$ sufficiently small. Moreover by $\mid w_0 - w^* \mid= R$ we have $v(w_0) = \tilde{v}(R) = 0 = \tilde{\Phi}(w_0)$, which yields together with (4.4):

$$\begin{aligned} \frac{\Phi(w_0) - \Phi(w)}{\mid w_0 - w \mid} &= \frac{\tilde{\Phi}(w_0) - \tilde{\Phi}(w)}{\mid w_0 - w \mid} = \frac{-\tilde{\Phi}(w)}{\mid w_0 - w \mid} \\ \geq \epsilon\,\frac{v(w)}{\mid w_0 - w \mid} &= \epsilon\,\frac{\tilde{v}(r) - \tilde{v}(R)}{\mid w_0 - w \mid} \qquad \forall\, w \in T_R(w^*). \end{aligned} \tag{4.5}$$

Now we set $\gamma(w) :=\mid \mathit{angle}(w - w_0, -w_0) \mid$ for $w \in B_R(w^*)$ and term $x := x(w) :=\mid w - w_0 \mid \sin\gamma(w)$ and $y := y(w) :=\mid w - w_0 \mid \cos\gamma(w)$ for $w \in K_{\hat{\delta},\frac{\pi}{4}}(w_0) := \{w \in B_{\hat{\delta}}(w_0) \cap B \mid \gamma(w) \in [0, \frac{\pi}{4}]\}$, with some $\hat{\delta} \in (0, \frac{R}{4})$. We have

$$r^2 \equiv\mid w - w^* \mid^2= x^2 + (R - y)^2 = x^2 + R^2 - 2Ry + y^2.$$

Combining this with $\mid w - w_0 \mid^2= x^2 + y^2$ we achieve:

$$\mid w - w_0 \mid^2= r^2 - R^2 + 2Ry$$

and therefore by $y =\mid w - w_0 \mid\ \cos\gamma(w)$:

$$\mid w - w_0 \mid= -\sqrt{r^2 - R^2\sin^2\gamma(w)} + R\,\cos\gamma(w),$$

$\forall\, w \in K_{\hat{\delta},\frac{\pi}{4}}(w_0)$. Together with the fact that $\frac{\partial}{\partial\gamma}(-\sqrt{r^2 - R^2\sin^2\gamma}+R\,\cos\gamma) \geq 0$, for any $r < R$ and $\gamma \in [0, \frac{\pi}{4}]$, we can conclude that

$$\mid w - w_0 \mid\leq -\sqrt{r^2 - R^2\sin^2\frac{\pi}{4}} + R\,\cos\frac{\pi}{4} = -\sqrt{r^2 - \frac{R^2}{2}} + \frac{R}{\sqrt{2}} \tag{4.6}$$

for any $w \in K_{\hat{\delta},\frac{\pi}{4}}(w_0)$. Now we claim that there exists some constant $c > 0$ and some $\hat{\delta} \in (0, \frac{R}{4})$ such that

$$\frac{R - r}{\mid w_0 - w \mid} > c \tag{4.7}$$

$\forall\, w \in K_{\hat{\delta},\frac{\pi}{4}}(w_0) \subset T_R(w^*)$. For suppose this assertion would be wrong, then for arbitrary null-sequences $\{c_n\}$ and $\{\delta_n\}$ there would have to exist points $w_n \in K_{\delta_n,\frac{\pi}{4}}(w_0)$ such that

$$\frac{q(r_n)}{Q(r_n)} := \frac{R - r_n}{-\sqrt{r_n^2 - \frac{R^2}{2}} + \frac{R}{\sqrt{2}}} \le \frac{R - r_n}{\mid w_0 - w_n \mid} \le c_n \qquad \forall\, n \in \mathbb{N}, \tag{4.8}$$

where we set $r_n :=\mid w_n - w^* \mid$ and used (4.6). On the other hand l'Hospital's rule yields

$$\lim_{r \nearrow R} \frac{q(r)}{Q(r)} = \lim_{r \nearrow R} \frac{q'(r)}{Q'(r)} = \lim_{r \nearrow R} \frac{-1}{-\frac{r}{\sqrt{r^2 - \frac{R^2}{2}}}} = \frac{1}{\sqrt{2}},$$

which contradicts (4.8) due to $c_n \searrow 0$ and $R > r_n > R - \delta_n \nearrow R$ and proves assertion (4.7). Now combining this with (4.5) we achieve for any $w \in K_{\hat{\delta},\frac{\pi}{4}}(w_0)$:

$$\frac{\Phi(w_0) - \Phi(w)}{\mid w_0 - w \mid} > \epsilon\, c\, \frac{\tilde{v}(r) - \tilde{v}(R)}{R - r} \longrightarrow -\epsilon\, c\, \tilde{v}_r(R) = \epsilon\, c\, 2\alpha\, R\, e^{-\alpha R^2} > 0,$$

for $r \nearrow R$, which implies in particular by $r =\mid w - w^* \mid \nearrow R$ for $\mid w - w_0 \mid \searrow 0$:

$$\liminf_{K_{\hat{\delta},\frac{\pi}{4}}(w_0) \ni w \to w_0} \frac{\Phi(w_0) - \Phi(w)}{\mid w_0 - w \mid} \ge 2\, \epsilon\, c\, \alpha\, R\, e^{-\alpha R^2}. \tag{4.9}$$

Thus there has to exist some $\delta > 0$ such that our assertion (4.2) holds for any $w \in K_{\delta,\frac{\pi}{4}}(w_0)$ and for $\sigma := \epsilon\, c\, \alpha\, R\, e^{-\alpha R^2} > 0$. Otherwise we could choose some null-sequence $\{\delta_n\} \subset (0, \hat{\delta})$ and would obtain for each n some point $w_n \in K_{\delta_n,\frac{\pi}{4}}(w_0)$ with

$$\frac{\Phi(w_0) - \Phi(w_n)}{\mid w_0 - w_n \mid} \le \epsilon\, c\, \alpha\, R\, e^{-\alpha R^2},$$

implying $\liminf_{K_{\hat{\delta},\frac{\pi}{4}}(w_0) \ni w \to w_0} \frac{\Phi(w_0) - \Phi(w)}{|w_0 - w|} \le \epsilon\, c\, \alpha\, R\, e^{-\alpha R^2}$ in contradiction to (4.9).

$\diamond$

Now using an idea of Sauvigny in [32], Hilfssatz 2, we are able to prove

Theorem 4.1 *If* Γ *is an extreme simple closed polygon then a minimal surface* $X \in \mathcal{M}(\Gamma)$ *does not possess any boundary branch points.*

Proof: We suppose the contrary, i.e. that there exists some boundary branch point $w_0 \in \partial B$ of $X \in \mathcal{M}(\Gamma)$. By the definition of a boundary branch point we infer that $| X_w(w) | \longrightarrow 0$ for $B \ni w \to w_0$, i.e. for any $\epsilon > 0$ there exists some $\tilde{\delta}(\epsilon) > 0$ such that

$$| DX(w) | < \epsilon \qquad \forall\, w \in B_{\tilde{\delta}}(w_0) \cap B. \tag{4.10}$$

Since we have $X(w_0) \in \Gamma \subset \partial K$ we can choose some supporting plane $E := \{x \in \mathbb{R}^3 \mid \langle x, e\rangle = c\}$ that touches the convex set K in $X(w_0)$, where the choice of the pair $e \in \mathbb{S}^2$ and $c \in \mathbb{R}$ is uniquely determined by the requirement that $K \subset H_{\leq} := \{x \in \mathbb{R}^3 \mid \langle x, e\rangle \leq c\}$. Hence, by $\mathrm{trace}(X\, |_{\partial B}) = \Gamma \subset \partial K \subset H_{\leq}$ we have then:

$$\langle X(w), e\rangle \leq c \qquad \forall\, w \in \partial B. \tag{4.11}$$

Now we consider the function $\Phi(\,\cdot\,) := \langle X(\,\cdot\,), e\rangle \in C^0(\bar{B}) \cap C^2(B)$ which is harmonic on B and satisfies therefore together with (4.11) and $X(w_0) \in E$:

$$\max_{\bar{B}} \Phi = \max_{\partial B} \Phi = \Phi(w_0) = c. \tag{4.12}$$

Now we claim that there even holds

$$\Phi(w) < \Phi(w_0) = c \qquad \forall\, w \in B, \tag{4.13}$$

otherwise, if Φ would assume its maximum in some point $w^* \in B$ Hopf's maximum principle for harmonic functions would yield that $\Phi \equiv$ *const.* $= c$ on $\bar{B}$, i.e. $\mathrm{image}(X) \subset E$, thus in particular $\mathrm{trace}(X\, |_{\partial B}) = \Gamma \subset E$, contradicting the requirement on Γ to be extreme. Hence, Φ satisfies all requirements of the above lemma which yields the existence of some constant $\sigma > 0$ such that there holds

$$\frac{\Phi(w_0) - \Phi(w)}{| w_0 - w |} > \sigma \qquad \forall\, w \in K_{\delta, \frac{\pi}{4}}(w_0), \tag{4.14}$$

for some sufficiently small chosen $\delta > 0$. Now by the mean value theorem we know that for any $w \in B$ there is some point $\xi(w) \in (w, w_0) := \{t\, w + (1-t)\, w_0 \mid t \in (0,1)\}$ such that

$$\Phi(w_0) - \Phi(w) = \langle \nabla\Phi(\xi(w)), w_0 - w\rangle \leq | \nabla\Phi(\xi(w)) |\, | w_0 - w | \,. \tag{4.15}$$

Furthermore we gain by (4.10) and $| \nabla\Phi | \leq | DX |$ that for any $\epsilon > 0$ there exists some $\tilde{\delta}(\epsilon) > 0$ such that

$$| \nabla\Phi(w) | < \epsilon \qquad \forall\, w \in B_{\tilde{\delta}}(w_0) \cap B. \tag{4.16}$$

Now we choose some $\epsilon \in (0, \sigma)$, set $\bar{\delta}(\epsilon) := \min\{\tilde{\delta}(\epsilon), \delta\} > 0$ and achieve by (4.14), (4.15), (4.16) and $| \xi(w) - w_0 |<| w - w_0 |$:

$$\sigma < \frac{\Phi(w_0) - \Phi(w)}{| w_0 - w |} \leq | \nabla\Phi(\xi(w)) | < \epsilon < \sigma \qquad \forall\, w \in K_{\bar{\delta}, \frac{\pi}{4}}(w_0),$$

which is a contradiction and proves our theorem.

◇

By the reasoning in (4.11) and (4.12) we obtain

Corollary 4.1 *Let Γ be an extreme simple closed polygon and let K denote the compact convex set whose boundary contains Γ. Then the image of every minimal surface $X \in \mathcal{M}(\Gamma)$ is contained in K.*

Proof: We choose an arbitrary point $Q \in \partial K$ and consider some supporting half space $H(Q)$ that contains K and whose boundary $\partial H(Q)$ touches the convex set K in the point Q, as in the above proof. Hence, by $\text{trace}(X \,|_{\partial B}) = \Gamma \subset \partial K \subset H(Q)$ we obtain as in (4.11) and (4.12) that $\text{image}(X) \subset H(Q)$. Now the assertion of the corollary follows from the well known fact that there holds $K = \bigcap_{Q \in \partial K} H(Q)$.

◇

Together with Theorem 1 on p. 175 in [7] we can derive the following important lemma, where we set $B_\delta(\bar{\tau}) := B^N_\delta(\bar{\tau}) \cap \mathbb{R}^N$ and fix some Γ as above:

Lemma 4.2 *Let $\bar{\tau} \in K(f)$, $k \in \{1, \ldots, N+3\}$ and $\delta > 0$ be fixed such that (2.8) holds for any $(w, \tau) \in (B_\delta(e^{i\bar{\tau}_k}) \cap B) \times B_\delta(\bar{\tau})$. Then we have the asymptotic expansions (2.9)*

$$X_w(w, \tau) = f^k_{j^*,0}(\tau)\, (w - e^{i\tau_k})^{\rho^k_{j^*}} + O(| w - e^{i\tau_k} |^{\rho^k_{j^*}+\epsilon}) \tag{4.17}$$

for $B \ni w \to e^{i\tau_k}$, simultanously for all $\tau \in B_\delta(\bar{\tau}) \cap K(f)$, i.e. one can not only assign the index $j^* \in \{1, 2\}$ of the leading summand of the asymptotic expansion (4.17) to each fixed $\bar{\tau} \in K(f)$ but even to an entire neighborhood $B_\delta(\bar{\tau}) \cap K(f)$ of this critical point within $K(f)$.

Proof: Firstly we know by Corollary 2.4 that the minimal surfaces $X(\,\cdot\,, \tau) \equiv \tilde{\psi}(\tau)$ coincide with $\psi(\tau)$, for $\tau \in B_\delta(\bar{\tau}) \cap K(f)$, and are therefore bounded by the extreme polygon Γ. Now it follows from Theorem 4.1 that $e^{i\tau_k}$ is not a branch point of $X(\,\cdot\,, \tau)$, for any fixed $k \in \{1, \ldots, N+3\}$. Thus first of all we obtain by (2.9) for any fixed τ an index $j^*(\tau) \in \{1, 2\}$ for which holds the asymptotic expansion (4.17) of $X_w(\,\cdot\,, \tau)$. Next since Corollary 4.1 guarantees that $\text{image}(X(\,\cdot\,, \tau))$ is contained in the convex compact set K, whose boundary contains Γ, for every $\tau \in B_\delta(\bar\tau) \cap K(f)$, Theorem 1 on p. 175 in [7] shows that the smaller angle $\beta_k \in (0,\pi)$ between the edges Γ_k and Γ_{k-1} of Γ determines the exponents $\rho^k_{j^*(\tau)}$ of the leading terms in (4.17) via the relation $\beta_k = \pi(1+\rho^k_{j^*(\tau)})$ simultanously for every τ. Hence, the function $\tau \mapsto j^*(\tau)$ has in fact the constant value 1 or 2 on $B_\delta(\bar\tau) \cap K(f)$.

$\diamond$

This insight yields the following important

Corollary 4.2 *Let $\bar\tau \in K(f)$ and $\delta > 0$ be fixed with the property that $X(\,\cdot\,,\tau)$ has no branch points on $\bar B$, i.e. $\kappa(\tau) = 0$, for $\tau \in B_\delta(\bar\tau) \cap K(f)$. Then there exists some $\bar\delta \in (0,\delta]$ and some constant C depending on Γ, $\bar\tau$ and $\bar\delta$ only such that there holds*

$$| \, (KE)^\tau(w) \, | \leq C \sum_{k=1}^{N+3} | \, w - e^{i\tau_k} \, |^{-2+\alpha} \qquad \forall\, w \in B, \tag{4.18}$$

for any $\tau \in K(f) \cap B_{\bar\delta}(\bar\tau)$.

Proof: We suppose the assertion of the lemma were wrong. Then there would have to exist some $k \in \{1,\ldots,N+3\}$ and some sequences $\{\tau^n\} \subset K(f)$ converging to $\bar\tau$ and $\{w^n\} \subset B$ such that

$$KE(w^n,\tau^n) \Big(\sum_{k=1}^{N+3} | \, w^n - e^{i\tau^n_k} \, |^{-2+\alpha} \Big)^{-1} \longrightarrow \infty \qquad \textit{for} \quad n \to \infty. \tag{4.19}$$

Now we substract some convergent subsequence $w^{n_j} \to \bar w \in \bar B$, rename it $\{w^n\}$ again and shall distinguish between two cases:
Case I) $\bar w \neq e^{i\bar\tau_k}$, for $k = 1,\ldots,N+3$, or Case II) $\bar w = e^{i\bar\tau_k}$, for some $k \in \{1,\ldots,N+3\}$. In the first case we obtain easily by (2.12), (2.37) and the assumption that any minimal surface $X(\,\cdot\,,\tau)$ is supposed to be free of branch points on $\bar B$ for any $\tau \in B_\delta(\bar\tau) \cap K(f)$:

$$\frac{KE(w^n,\tau^n)}{\sum_{k=1}^{N+3} | \, w^n - e^{i\tau^n_k} \, |^{-2+\alpha}} \longrightarrow \frac{KE(\bar w,\bar\tau)}{\sum_{k=1}^{N+3} | \, \bar w - e^{i\bar\tau_k} \, |^{-2+\alpha}} \qquad \textit{for} \quad n \to \infty,$$

in contradiction to (4.19). Now we consider the second case. We can choose some $\tilde\delta \in (0,\delta]$ that small such that there holds (2.6) for any $w \in B_{\tilde\delta}(e^{i\bar\tau_k}) \cap B$ and any $\tau \in B_{\tilde\delta}(\bar\tau)$. By a lengthy but easy computation one obtains by (2.6), the derived representation of $X_{ww}(\,\cdot\,,\tau)$ and (2.34):

$$\frac{| \, \langle \xi(w,\tau), X_{ww}(w,\tau) \rangle \, |}{| \, X_w(w,\tau) \, |} \leq \frac{b^k(w,\tau) \; | \, w - e^{i\tau_k} \, |^{-1+\epsilon}}{| \, f^k_{j^*}(w,\tau) + \sum_{j=j^*+1}^{p_k} f^k_j(w,\tau) \, (w - e^{i\tau_k})^{\rho^k_j - \rho^k_{j^*}} \, |^3}, \tag{4.20}$$

for some continuous bounded function b^k on $B_{\tilde{\delta}}(e^{i\bar{\tau}_k}) \times B_{\tilde{\delta}}(\bar{\tau})$ and $\epsilon := \frac{\alpha}{2} > 0$ (see (2.39)), where we made decisive use of Lemma 4.2 guaranteeing the local constancy of the assignment $\tau \mapsto j^*(\tau)$. Moreover we may simplify our hypothesis (4.19) by $w^n \to e^{i\bar{\tau}_k} \leftarrow e^{i\tau_k^n}$ into the form

$$KE(w^n, \tau^n) \mid w^n - e^{i\tau_k^n} \mid^{2-\alpha} \longrightarrow \infty \qquad \textit{for} \quad n \to \infty.$$

Together with identity (2.37), (4.20) and the boundedness of b^k we arrive at

$$f^k_{j^*}(w^n, \tau^n) + \sum_{j=j^*+1}^{p_k} f^k_j(w^n, \tau^n)\,(w^n - e^{i\tau_k^n})^{\rho^k_j - \rho^k_{j^*}} \longrightarrow 0 \qquad \textit{for} \quad n \to \infty.$$

Now taking into account that $w^n \to e^{i\bar{\tau}_k} \leftarrow e^{i\tau_k^n}$ and the continuity of the functions f^k_j on $B_{\tilde{\delta}}(e^{i\bar{\tau}_k}) \times B_{\tilde{\delta}}(\bar{\tau})$ we arrive at $f^k_{j^*}(w^n, \tau^n) \to 0 = f^k_{j^*}(e^{i\bar{\tau}_k}, \bar{\tau}) = f^k_{j^*,0}(\bar{\tau})$. But this contradicts the fact that $f^k_{j^*,0}(\bar{\tau}) \neq 0$ by the definition of j^* (see below (2.8)) and thus proves the assertion of the lemma.

◇

Moreover there holds for an arbitrary simple closed polygon Γ:

Lemma 4.3 *Let $X(\,\cdot\,, \tau) \in \tilde{\mathcal{M}}(\Gamma)$ be a minimal surface whose boundary values are not monotonic on some arc $(e^{i\tau_k}, e^{i\tau_{k+1}}) \subset \mathbb{S}^1$, for some $k = 1, \dots, N+3$. Then there exists some angle $\theta \in (\tau_k, \tau_{k+1})$ for which $e^{i\theta}$ is a boundary branch point of $X(\,\cdot\,, \tau)$.*

Proof: We abbreviate $X := X(\,\cdot\,, \tau)$. Since point (vi) of Theorem 2.1 guarantees that the derivatives $D_{(u,v)}X$ can be continued continuously onto $\bar{B} \setminus \{e^{i\tau_k}\}_{k=1,\dots,N+3}$ the conformality relations of X still hold especially on the open arc $(e^{i\tau_k}, e^{i\tau_{k+1}}) \subset \mathbb{S}^1$. Now as the boundary values $X \mid_{\partial B}$ are required to be not monotonic on $(e^{i\tau_k}, e^{i\tau_{k+1}})$ there has to exist some angle $\theta \in (\tau_k, \tau_{k+1})$ with $\frac{d}{d\varphi} X(e^{i\varphi}) \mid_{\varphi=\theta} = 0$, and thus we achieve as in the proof of Corollary 2.3 that $e^{i\theta}$ is a boundary branch point of X.

◇

Corollary 4.3 *Any minimal surface $X \in \tilde{\mathcal{M}}(\Gamma) \setminus \mathcal{M}(\Gamma)$ possesses a boundary branch point.*

Proof: Firstly we know by Lemma 2.1 that for any fixed minimal surface $X \in \tilde{\mathcal{M}}(\Gamma)$ there exists some $\tau \in T$ with $X = X(\,\cdot\,, \tau)$ (even $\tau \in K(\tilde{f})$ by Theorem 2.1, (v)). X is required not to be contained in $\mathcal{C}^*(\Gamma)$, which implies by $X \in \tilde{\mathcal{U}}(\tau) \cap H^{1,2}(B, \mathbb{R}^3)$ that X is required not to have weakly monotonic boundary values. Now noting that we have especially $X(e^{i\tau_j}) = P_j$, for $j = 1, \dots, N+3$, this shows that the requirement of the above lemma is satisfied, which yields our assertion immediately.

◇

Now combining Lemma 4.3 with the ideas of the proof of Theorem 4.1 we can also derive for extreme simple closed polygons Γ:

Corollary 4.4 *Let Γ be some extreme simple closed polygon. If a minimal surface $X \in \tilde{\mathcal{M}}(\Gamma)$ satisfies $2\,\kappa(X) = 1$ then it is already free of branch points on $\bar{B}$.*

Proof: By the requirement $2\,\kappa(X) = 1$ the surface X can have at most one boundary branch point $w_0 \in \partial B$, i.e. which satisfies (4.10). Now by $X \in \tilde{\mathcal{M}}(\Gamma)$ we have $X(w_0) \in \Gamma_j := \{P_j + t\,(P_{j+1} - P_j) \mid t \in \mathbb{R}\}$ for some $j \in \{1, \dots, N+3\}$. Hence, we can choose some supporting plane $E := \{x \in \mathbb{R}^3 \mid \langle x, e\rangle = c\}$ of the convex set K which touches K in the point $X(w_0)$ and contains the line Γ_j, where again the choice of the pair $e \in \mathbb{S}^2$ and $c \in \mathbb{R}$ is uniquely determined by the requirement that K is contained in the half space $H_{\leq} := \{x \in \mathbb{R}^3 \mid \langle x, e\rangle \leq c\}$. Moreover we know by Lemma 2.1 that there exists some $\tau \in T$ such that $X = X(\,\cdot\,, \tau) \in \tilde{\mathcal{U}}(\tau)$, which implies firstly that $\text{trace}(X(e^{i\,\cdot}\,) \mid_{[\tau_j, \tau_{j+1}]}) \subset \Gamma_j \subset E$. Moreover since $X = X(\,\cdot\,, \tau)$ does not possess any further branch points we can infer from Lemma 4.3 that $X(e^{i\varphi}) \in \Gamma$ for any $\varphi \in [0, 2\pi] \setminus (\tau_j, \tau_{j+1})$. Otherwise there would have to be some intervall $[\tau_k, \tau_{k+1}]$, for some $k \neq j$, such that $\text{trace}(X(e^{i\,\cdot}\,) \mid_{[\tau_k, \tau_{k+1}]}) \not\subset [P_k, P_{k+1}]$ on account of the boundary conditions imposed on surfaces in the class $\tilde{\mathcal{U}}(\tau)$. Now recalling especially the required continuity of the boundary values $X \mid_{\partial B}$ and $X(e^{i\tau_l}) = P_l$, for $l = 1, \dots, N+3$, this would imply in particular that $X \mid_{\partial B}$ is not monotonic on the respective arc $(e^{i\tau_k}, e^{i\tau_{k+1}}) \subset \mathbb{S}^1$ and thus the existence of a further boundary branch point on $(e^{i\tau_k}, e^{i\tau_{k+1}})$ due to Lemma 4.3. Hence, together with $\Gamma \subset \partial K \subset H_{\leq}$ we infer that $\text{trace}(X \mid_{\partial B}) \subset \Gamma \cup \Gamma_j \subset H_{\leq}$. Thus we achieve as in the proof of Theorem 4.1 that

$$\Phi(w) := \langle X(w), e\rangle \leq c \qquad \forall\, w \in \partial B$$

and together with the harmonicity of Φ on B and $X(w_0) \in E$:

$$\max_{\bar{B}} \Phi = \max_{\partial B} \Phi = \Phi(w_0) = c.$$

Moreover we state that Φ inherits its continuity on $\bar{B}$ from $X \in \tilde{\mathcal{M}}(\Gamma)$. Next we note that in our situation the trace of $X \mid_{\partial B}$ still contains Γ. Thus as in the proof of Theorem 4.1 Hopf's maximum principle applied to Φ yields that Φ cannot attain its maximum in some point of B, as otherwise we would obtain $\Phi \equiv c$ on $\bar{B}$, implying especially $\Gamma \subset \text{trace}(X \mid_{\partial B}) \subset E$ in contradiction to the required extremeness of Γ. Hence, having achieved inequality (4.13) again we can apply Lemma 4.1 to Φ in order to derive a contradiction to (4.10) resp. (4.16), thus to exclude the assumed existence of the boundary branch point w_0 of X exactly as in the ending of the proof of Theorem 4.1, which proves our assertion.

$\diamond$

Chapter 5

Mollified Green functions on $\bar{B}$

As in [9] we consider the bounded, positive definite bilinear form

$$a(\phi, \psi) := \int_B \nabla\phi \cdot \nabla\psi \, dw \tag{5.1}$$

on $\mathring{H}^{1,2}(B) \times \mathring{H}^{1,2}(B)$. Moreover for fixed $\rho > 0$ and $y \in B$ the map $\phi \longmapsto \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \phi \, dw$ is a linear functional on $\mathring{H}^{1,2}(B)$. Hence, from the Lax-Milgram theorem we can infer the unique existence of a function $G^\rho(\,\cdot\,, y) \in \mathring{H}^{1,2}(B)$ such that there holds

$$a(G^\rho(\,\cdot\,, y), \phi) = \frac{1}{\mid B_\rho(y) \mid} \int_{B_\rho(y)} \phi \, dw, \tag{5.2}$$

for any $\phi \in \mathring{H}^{1,2}(B)$. As in [9], p. 5, one can easily show that $G^\rho \geq 0$ a.e. on B.

Definition 5.1 *Let $\Omega \subset B$ be some domain and f some $\mathcal{L}^2$-measurable function on Ω. Then the weak $L^p(\Omega)$-norm of f, for $p \geq 1$, is defined as*

$$\parallel f \parallel_{L^p_w(\Omega)} := \sup_{t>0} t \, \mathcal{L}^2(\{w \in \Omega \mid \mid f(w) \mid > t\})^{\frac{1}{p}}.$$

One can easily prove the following estimate due to Stampacchia:

$$\parallel f \parallel_{L^{p-\epsilon}(\Omega)} \leq \left(\frac{p}{\epsilon}\right)^{\frac{1}{p-\epsilon}} \mathcal{L}^2(\Omega)^{\frac{\epsilon}{p(p-\epsilon)}} \parallel f \parallel_{L^p_w(\Omega)}, \tag{5.3}$$

$\forall \, \epsilon \in (0, p-1)$. Now similarly as in [9] we prove

Proposition 5.1 *Let $y \in B$ be some arbitrarily fixed point. Then there holds for $G^\rho := G^\rho(\,\cdot\,, y)$ and any $p \in [1, \infty)$:*

$$\parallel G^\rho \parallel_{L^p(B)} \leq \text{const.}(p) \qquad \forall \, \rho > 0. \tag{5.4}$$

Proof: We fix some $\rho > 0$ and abbreviate $B(t) := \{w \in B \mid G^\rho(w) > t\}$. Firstly we consider the function

$$v_t(s) := \begin{cases} 0 & : \quad 0 \le s \le t \\ t^{-1} - s^{-1} & : \quad s > t, \end{cases}$$

for some fixed $t > 0$ and test (5.2) with the composition $\phi := v_t \circ G^\rho \in \mathring{H}^{1,2}(B)$, which yields:

$$\int_{B(t)} \Big(\frac{\mid \nabla G^\rho \mid}{G^\rho}\Big)^2 dw = a(G^\rho, \phi) = \frac{1}{\mid B_\rho(y) \mid} \int_{B_\rho(y)} v_t \circ G^\rho \, dw \le t^{-1}. \quad (5.5)$$

Next we consider the function

$$y_t(s) := \begin{cases} 0 & : \quad 0 \le s \le t \\ \log \frac{s}{t} & : \quad s > t, \end{cases}$$

for some fixed $t > 0$ and apply the continuity of the embedding $H^{1,2}(B) \hookrightarrow L^q(B)$, for any $q \ge 1$, and the Poincaré inequality to the composition $\phi := y_t \circ G^\rho \in \mathring{H}^{1,2}(B)$ in combination with estimate (5.5):

$$\parallel \phi \parallel^2_{L^q(B(t))} \le C(q) \int_{B(t)} \mid \nabla\phi \mid^2 \, dw = C(q) \int_{B(t)} \Big(\frac{\mid \nabla G^\rho \mid}{G^\rho}\Big)^2 dw \le \frac{C(q)}{t},$$

$\forall\, t > 0$. Thus due to $\frac{G^\rho}{t} > 2$ on $B(2t)$ we can estimate for any $q \ge 2$:

$$\begin{aligned}(\log 2)^2 \ \mid B(2t) \mid^{\frac{2}{q}} = \Big(\int_{B(2t)} (\log 2)^q \, dw\Big)^{\frac{2}{q}} < \Big(\int_{B(2t)} \Big(\log \frac{G^\rho}{t}\Big)^q dw\Big)^{\frac{2}{q}} \\ = \parallel \phi \parallel^2_{L^q(B(2t))} \le \frac{C(q)}{2t},\end{aligned}$$

$\forall\, t > 0$. Hence, setting $\sigma := 2t$ we achieved $\sigma \mid B(\sigma) \mid^{\frac{2}{q}} < const.(q)$, $\forall\, \sigma > 0$, thus

$$\parallel G^\rho \parallel_{L^{\frac{q}{2}}_w(B)} < const.(q), \quad (5.6)$$

$\forall\, q \ge 2$. Hence, together with estimate (5.3) we achieve the assertion of the proposition by setting $p := \frac{q}{2} - \epsilon$ for some fixed $\epsilon > 0$.

$\diamond$

Next we prove as in [9]:

Proposition 5.2 *Let $y \in B$ be some arbitrarily fixed point. Then there holds for $G^\rho := G^\rho(\,\cdot\,, y)$ and any $s \in (1,2)$:*

$$\parallel G^\rho \parallel_{H^{1,s}(B)} \le \text{const.}(s) \qquad \forall\, \rho > 0. \quad (5.7)$$

Proof: Firstly we consider the function

$$f_\alpha(s) := \begin{cases} 0 & : \ s \le 0 \\ 1-(1+s)^{\alpha-1} & : \ s > 0, \end{cases}$$

for some fixed $\alpha \in (0,1)$ and test (5.2) with the composition $\phi := f_\alpha \circ G^\rho$, for some chosen $\rho > 0$, which yields

$$\begin{aligned} \int_B \mid \nabla G^\rho \mid^2 (1+G^\rho)^{\alpha-2}\, dw &= \int_B \nabla G^\rho \cdot \frac{1}{1-\alpha} \nabla\phi\, dw \\ &= \frac{1}{(1-\alpha)\ \mid B_\rho(y) \mid} \int_{B_\rho(y)} \phi\, dw \\ \le \frac{1}{(1-\alpha)\ \mid B_\rho(y) \mid} \int_{B_\rho(y)} 1-(1+G^\rho)^{\alpha-1}\, dw &\le \frac{1}{1-\alpha}, \end{aligned} \tag{5.8}$$

$\forall\, \rho > 0$. Now we choose some $s \in (1,2)$ arbitrarily and set $r := \frac{2}{s} > 1$, $q := \frac{2}{2-s}$, such that $\frac{1}{r} + \frac{1}{q} = 1$, and $\alpha := \frac{-2+2s}{s} \in (0,1)$. Combining (5.8) with Hölder's and Minkowski's inequality and with Prop. 5.1 applied to $p := \frac{2}{2-s} > 2$ we achieve:

$$\begin{aligned} \| \nabla G^\rho \|_{L^s(B)}^s &= \int_B \mid \nabla G^\rho \mid^s (1+G^\rho)^{\frac{s}{2}(\alpha-2)} (1+G^\rho)^{\frac{s}{2}(2-\alpha)}\, dw \\ &\le \Big(\int_B \mid \nabla G^\rho \mid^2 (1+G^\rho)^{\alpha-2}\, dw \Big)^{\frac{s}{2}} \Big(\int_B (1+G^\rho)^{(2-\alpha)\frac{s}{2}\frac{2}{2-s}}\, dw \Big)^{\frac{2-s}{2}} \\ \le (1-\alpha)^{-\frac{s}{2}} \Big(\int_B (1+G^\rho)^{\frac{2}{2-s}}\, dw \Big)^{\frac{2-s}{2}} &\le \Big(\frac{2-s}{s} \Big)^{-\frac{s}{2}} (\mid B \mid^{\frac{2-s}{2}} + \| G^\rho \|_{L^{\frac{2}{2-s}}(B)}) \\ &\le \Big(\frac{2-s}{s} \Big)^{-\frac{s}{2}} \Big(\pi^{\frac{2-s}{2}} + const.\Big(\frac{2}{2-s} \Big) \Big) = const.(s), \end{aligned}$$

$\forall\, \rho > 0$, where we used that $2-\alpha = \frac{2}{s}$ and $1-\alpha = \frac{2-s}{s}$. Hence, together with a suitable Poincaré inequality (see [1], p. 224) we obtain the assertion of the proposition.

$\diamond$

Now let $\{\rho_j\}$ be some null-sequence and $s_k \nearrow 2$. On account of Prop. 5.2 we can successively extract weakly convergent subsequences of $\{G^{\rho_j}(\,\cdot\,,y)\}$ in $\mathring{H}^{1,s_k}(B)$ for each $k \in \mathbb{N}$. Choosing a diagonal sequence we consequently obtain some subsequence $\{G^{\rho_{j_k}}(\,\cdot\,,y)\}$ of $\{G^{\rho_j}(\,\cdot\,,y)\}$ which satisfies

$$G^{\rho_{j_k}}(\,\cdot\,,y) \rightharpoonup G(\,\cdot\,,y) \qquad \text{weakly in } \mathring{H}^{1,s}(B), \tag{5.9}$$

for some function $G(\,\cdot\,,y) \in \mathring{H}^{1,s}(B)$ and any $s \in (1,2)$. Now together with Sobolev's and Rellich's embedding theorems we conclude from this that

$$G^{\rho_{j_k}}(\,\cdot\,,y) \longrightarrow G(\,\cdot\,,y) \qquad \textit{in } L^p(B), \tag{5.10}$$

for any $p \in [1, \infty)$. Thus by the lower semicontinuity of $\| \cdot \|_{L^p(B)}$ and (5.4) we have:

$$\| G(\cdot, y) \|_{L^p(B)} \leq \liminf_{k \to \infty} \| G^{\rho_{j_k}}(\cdot, y) \|_{L^p(B)} \leq \mathit{const.}(p) \qquad \forall\, y \in B, \tag{5.11}$$

$\forall\, p \in (1, \infty)$, and by the weak lower semicontinuity of $\| \cdot \|_{H^{1,s}(B)}$ and (5.7):

$$\| G(\cdot, y) \|_{H^{1,s}(B)} \leq \liminf_{k \to \infty} \| G^{\rho_{j_k}}(\cdot, y) \|_{H^{1,s}(B)} \leq \mathit{const.}(s) \qquad \forall\, y \in B, \tag{5.12}$$

$\forall\, s \in (1, 2)$. Next using (5.2) we derive a Cacciopoli type inequality for $G^\rho := G^\rho(\cdot, y)$, for any fixed $y \in B$:

Proposition 5.3 *For any $R > 0$ there holds:*

$$\int_{B \setminus \overline{B_R(y)}} |\, \nabla G^\rho \,|^2 \; dw \leq \mathrm{const.}\, \frac{1}{R^2} \int_{T_R(y)} |\, G^\rho \,|^2 \; dw, \tag{5.13}$$

$\forall\, \rho \in (0, \frac{R}{2}]$, *where we set* $T_R(y) := B_R(y) \setminus \overline{B_{\frac{R}{2}}(y)}$.

Proof: We test (5.2) with $\eta^2 G^\rho$, where we require $\eta \in C_c^\infty(\mathbb{R}^2 \setminus \overline{B_{\frac{R}{2}}(y)})$ with $\eta \equiv 1$ on $\bar{B} \setminus B_R(y)$, $|\, \nabla \eta \,| \leq \frac{C}{R}$ on $T_R(y)$ and $0 \leq \eta \leq 1$ on $\mathbb{R}^2$. Then we infer due to $\eta \equiv 0$ on $\overline{B_{\frac{R}{2}}(y)}$ together with Cauchy-Schwarz' inequality, for $\rho \leq \frac{R}{2}$:

$$\begin{aligned} & \int_{B \setminus \overline{B_{\frac{R}{2}}(y)}} |\, \nabla G^\rho \,|^2 \; \eta^2 \, dw \\ = -2 & \int_{B \setminus \overline{B_{\frac{R}{2}}(y)}} \nabla G^\rho \cdot \nabla \eta \; \eta \, G^\rho \, dw + \frac{1}{|\, B_\rho(y) \,|} \int_{B_\rho(y)} \eta^2 G^\rho \, dw \\ \leq & \; \frac{1}{2} \int_{B \setminus \overline{B_{\frac{R}{2}}(y)}} |\, \nabla G^\rho \,|^2 \; \eta^2 \, dw + \frac{2\, C^2}{R^2} \int_{T_R(y)} |\, G^\rho \,|^2 \; dw. \end{aligned}$$

Hence, together with $\eta \equiv 1$ on $\bar{B} \setminus B_R(y)$ and $0 \leq \eta \leq 1$ absorbing yields:

$$\begin{aligned} \int_{B \setminus \overline{B_R(y)}} |\, \nabla G^\rho \,|^2 \; dw & \leq \int_{B \setminus \overline{B_{\frac{R}{2}}(y)}} |\, \nabla G^\rho \,|^2 \; \eta^2 \, dw \\ & \leq \frac{4\, C^2}{R^2} \int_{T_R(y)} |\, G^\rho \,|^2 \; dw, \qquad \forall\, \rho \leq \frac{R}{2}. \end{aligned}$$

◇

Next we prove for $G^\rho := G^\rho(\cdot, y)$, for any fixed $y \in B$:

Proposition 5.4 *For an arbitrary $R \in (0, \frac{1-|y|}{2})$ there hold the estimates*

$$G^\rho(w) \le C(q)\, R^{-\frac{4}{q}} \qquad \forall\, w \in T_R(y), \tag{5.14}$$

$\forall\, \rho < \frac{R}{4}$ and $\forall\, q > 2$,

$$\int_{B \setminus \overline{B_R(y)}} |\, \nabla G^\rho \,|^2 \; dw \le C(q)\, R^{-\frac{8}{q}}, \tag{5.15}$$

$\forall\, \rho < \frac{R}{4}$ and $\forall\, q > 2$. Moreover we derive $G(\,\cdot\,, y) \in H^{1,2}(B \setminus \overline{B_R(y)})$ with

$$\int_{B \setminus \overline{B_R(y)}} |\, \nabla_w G(\,\cdot\,, y)\,|^2 \; dw \le C(q)\, R^{-\frac{8}{q}}, \tag{5.16}$$

$\forall\, q > 2$ and for any $R \in (0, \frac{1-|y|}{2})$.

Proof: We fix some arbitrary $R \in (0, \frac{1-|y|}{2})$, $w \in T_R(y)$ and $\rho < \frac{R}{4}$ and note that $B_{\frac{R}{4}}(w) \subset\subset B \setminus \overline{B_\rho(y)}$. Due to $a(G^\rho, \phi) = 0 \;\; \forall\, \phi \in C_c^\infty(B \setminus \overline{B_\rho(y)})$ by (5.2) we may apply a Moser-Harnack type inequality to $G^\rho \ge 0$ on $\Omega := B \setminus \overline{B_\rho(y)}$ (see Lemma 1.2 in [9]) yielding:

$$\begin{aligned} (G^\rho(w))^\alpha \le \sup_{B_{\frac{R}{8}}(w)} (G^\rho)^\alpha &\le C(\alpha)\, \frac{1}{|\, B_{\frac{R}{4}}(w)\,|} \int_{B_{\frac{R}{4}}(w)} |\, G^\rho \,|^\alpha \; dw \\ &= C(\alpha)\, R^{-2} \parallel G^\rho \parallel^\alpha_{L^\alpha(B_{\frac{R}{4}}(w))} . \end{aligned}$$

Now we fix some $q > 2$ arbitrarily and set $\alpha := \frac{q}{2} - \epsilon$ for $\epsilon := \frac{q}{4} - \frac{1}{2} > 0$ such that $\alpha = \frac{q}{4} + \frac{1}{2} > 1$ and estimate by (5.3) and (5.6):

$$\begin{aligned} R^{-2} \parallel G^\rho \parallel^\alpha_{L^\alpha(B_{\frac{R}{4}}(w))} &\le R^{-2} \frac{q}{2\epsilon} \,|\, B_{\frac{R}{4}}(w)\,|^{\frac{2\epsilon}{q}} \parallel G^\rho \parallel^{\frac{q}{4}+\frac{1}{2}}_{L_w^{\frac{q}{2}}(B_{\frac{R}{4}}(w))} \\ &\le \mathit{const.}(q)\, R^{-2+2\frac{2\epsilon}{q}} = \mathit{const.}(q)\, R^{-\frac{4}{q}\alpha}, \end{aligned}$$

for we have $-2 + 2\frac{2\epsilon}{q} = 2\left(\frac{\epsilon}{q/2} - 1\right) = 2\left(-\frac{\alpha}{q/2}\right) = -\frac{4}{q}\alpha$. Hence, we achieved (5.14) for any $w \in T_R(y)$ and $\forall\, q > 2$. Now we combine this with Cacciopoli's inequality (5.13) and obtain

$$\int_{B \setminus \overline{B_R(y)}} |\, \nabla G^\rho \,|^2 \; dw \le \frac{C}{R^2} \int_{T_R(y)} |\, G^\rho \,|^2 \; dw \le \frac{C(q)}{R^2} R^2 R^{-\frac{8}{q}} = C(q)\, R^{-\frac{8}{q}},$$

$\forall\, \rho < \frac{R}{4}$ and $\forall\, q > 2$, hence we achieved (5.15). Together with (5.4) we have

$$\parallel G^\rho \parallel_{H^{1,2}(B \setminus \overline{B_R(y)})} \le \mathit{const.}(2) + C(q)\, R^{-\frac{8}{q}},$$

which in combination with (5.9) implies the existence of a null sequence $\{\rho_j\}$ such that

$$G^{\rho_j} \rightharpoonup G(\,\cdot\,, y) \qquad \text{in } \; H^{1,2}(B \setminus \overline{B_R(y)}),$$

and therefore estimate (5.16), $\forall\, q > 2$ and $\forall\, R \in (0, \frac{1-|y|}{2})$, on account of the weak lower semicontinuity of the Dirichlet integral.

$\diamond$

Chapter 6

The classical Green function on $\bar{B}$

In the sequel we will show that $G(\,\cdot\,, y)$ coincides with the function

$$\tilde{G}(w,y) := \frac{1}{2\pi}\log\Big(\frac{\mid 1-\bar{w}y \mid}{\mid w-y \mid}\Big), \tag{6.1}$$

which we shall consider on $(\bar{B}\times\bar{B})\setminus\Lambda$ with $\Lambda := \{(w,y)\in\bar{B}\times\bar{B} \mid w=y\}$ and will turn out to be the classical symmetric Green function w. r. to the domain B. We note that $\frac{y-w}{\bar{w}y-1} =: \kappa_w(y)$ is a conformal automorphism of the disc $\bar{B}$. We firstly need

Proposition 6.1 *$\tilde{G}(w,y)$ is symmetric in y and w on $(\bar{B}\times\bar{B})\setminus\Lambda$ and we derive for any fixed $w^*\in B$ and $y^*\in\bar{B}\setminus\{w^*\}$*

$$\frac{\partial^s}{\partial y^s}\tilde{G}(w^*,y^*) = \frac{\partial^s}{\partial w^s}\tilde{G}(y^*,w^*), \tag{6.2}$$

$\forall\, s=(s_1,s_2)\in\{0,1,2\}^2$, with $\mid s \mid := s_1+s_2\le 2$, where

$$\partial_y\tilde{G}(w,y) = \frac{1}{4\pi}\Big(\frac{1}{w-y} - \frac{\bar{w}}{1-\bar{w}y}\Big), \tag{6.3}$$

$$\partial_{yy}\tilde{G}(w,y) = \frac{1}{4\pi}\Big(\frac{1}{(w-y)^2} - \frac{\bar{w}^2}{(1-\bar{w}y)^2}\Big) \tag{6.4}$$

$\forall\, y\in\bar{B}\setminus\{w\}$ and for any fixed $w\in B$. Moreover for any $w\in B$ there holds $\tilde{G}(w,\,\cdot\,)\in L^p(B)$, $\forall\, p\in[1,\infty)$, and we have $\tilde{G}(w,\,\cdot\,)\equiv 0$ on ∂B for any $w\in B$. Furthermore we derive the estimates

$$\mid \nabla_y\tilde{G}(w,y) \mid < \frac{1}{\pi}\frac{1}{\mid y-w \mid}, \tag{6.5}$$

$\forall\, y\in\bar{B}\setminus\{w\}$ and any fixed $w\in B$, thus $\tilde{G}(w,\,\cdot\,)\in \mathring{H}^{1,s}(B)$, $\forall\, s<2$, with

$$\parallel \nabla_y\tilde{G}(w,\,\cdot\,) \parallel_{L^s(B)} < \text{const.}(s) \qquad \forall\, w\in B, \tag{6.6}$$

and also

$$\| \nabla_w \tilde{G}(w, \cdot\,) \|_{L^s(B)} < \text{const.}(s) \qquad \forall\, w \in B, \tag{6.7}$$

and

$$| D_y^2 \tilde{G}(w,y) | \leq 4 \mid \partial_{yy} \tilde{G}(w,y) | < \frac{2}{\pi} \frac{1}{| y-w |^2}, \tag{6.8}$$

$\forall\, y \in \bar{B} \setminus \{w\}$ *and any fixed* $w \in B$*. Moreover there holds*

$$\triangle_y \tilde{G}(w,y) = \frac{1}{2\pi} \det D_y \kappa_w(y) \, \triangle \Big(\log \frac{1}{|\cdot|} \Big) \circ \kappa_w(y) = 0 \tag{6.9}$$

$\forall\, y \in \bar{B} \setminus \{w\}$ *and any* $w \in B$*, and*

$$\partial_\nu \tilde{G}(w,y) = \frac{1}{2\pi} \frac{| w |^2 - 1}{| y-w |^2} \qquad \forall\, y \in \partial B. \tag{6.10}$$

Proof: The symmetry of $\tilde{G}(\,\cdot\,,\,\cdot\,)$ on $(\bar{B} \times \bar{B}) \setminus \Lambda$ can be easily verified, which implies

$$\frac{\partial}{\partial y_i} \tilde{G}(w^*, y^*) = \lim_{\epsilon \to 0} \frac{\tilde{G}(w^*, y^* + \epsilon e_i) - \tilde{G}(w^*, y^*)}{\epsilon}$$
$$= \lim_{\epsilon \to 0} \frac{\tilde{G}(y^* + \epsilon e_i, w^*) - \tilde{G}(y^*, w^*)}{\epsilon} = \frac{\partial}{\partial w_i} \tilde{G}(y^*, w^*)$$

for any fixed $w^* \in B$, $y^* \in \bar{B} \setminus \{w^*\}$ and $i = 1, 2$. Similarily one can derive this symmetry for the derivatives of $\tilde{G}(w,y)$ of second order. Next $\tilde{G}(w, \cdot\,) \in L^p(B)$, $\forall\, p \in [1, \infty)$, is an easy consequence of the transformation theorem on account of $\tilde{G}(w,y) = \frac{1}{2\pi}\big(\log \frac{1}{|\cdot|}\big) \circ \kappa_w(y)$, where $\kappa_w(y) := \frac{y-w}{\bar{w}y-1}$ is a conformal automorphism of the disc $\bar{B}$, with

$$\det D_y \kappa_w(y) = | \partial_y \kappa_w(y) |^2 = \frac{(1 - | w |^2)^2}{| \bar{w}y - 1 |^4} > \frac{(1 - | w |^2)^2}{2^4} \tag{6.11}$$

$\forall\, y \in \bar{B}$ and $\forall\, w \in B$, and therefore

$$\int_B \tilde{G}(w,y)^p \, dy = \frac{1}{(2\pi)^p} \int_B \Big(\log \frac{1}{|\cdot|} \Big)^p \circ \kappa_w(y) \, dy$$
$$< \frac{2^4}{(2\pi)^p \, (1 - | w |^2)^2} \int_B \Big(\log \frac{1}{|\cdot|} \Big)^p \circ \kappa_w(y) \, \det D_y \kappa_w(y) \, dy$$
$$= \frac{16}{(2\pi)^p \, (1 - | w |^2)^2} \int_B \Big(\log \frac{1}{| z |} \Big)^p dz$$
$$= \frac{16 \, (2\pi)^{1-p}}{(1 - | w |^2)^2} \int_0^1 r \Big(\log \frac{1}{r} \Big)^p dr < \infty$$

$\forall\, p \geq 1$ and $\forall\, w \in B$. Furthermore we have for $y \in \partial B$:

$$| \bar{w}y - 1 | = | y | \, | 1 - \bar{y}w | = | y - y\bar{y}w | = | y - w |, \tag{6.12}$$

hence, indeed $\tilde{G}(w,\cdot\,)\equiv 0$ on ∂B, $\forall\, w\in B$. Next we compute:

$$\frac{|\,1-\bar{w}y\,|^2}{|\,y-w\,|^2}=\frac{1-\bar{w}y}{y-w}\,\frac{1-w\bar{y}}{\bar{y}-\bar{w}}=\frac{1+|\,wy\,|^2-w\bar{y}-\bar{w}y}{|\,y\,|^2+|\,w\,|^2-y\bar{w}-w\bar{y}},$$

and therefore

$$\partial_y\Big(\frac{|\,1-\bar{w}y\,|}{|\,y-w\,|}\Big)=\frac{1}{2}\frac{|\,y-w\,|}{|\,1-\bar{w}y\,|}\cdot\frac{|\,y-w\,|^2\;(\bar{y}\,|\,w\,|^2-\bar{w})-|\,1-\bar{w}y\,|^2\;(\bar{y}-\bar{w})}{|\,y-w\,|^4}.$$

Furthermore one easily computes that

$$\begin{aligned}&|\,y-w\,|^2\;(\bar{y}\,|\,w\,|^2-\bar{w})-|\,1-\bar{w}y\,|^2\;(\bar{y}-\bar{w})\\&\qquad=(\overline{y-w})\,((1-\bar{y}w)\,|\,w\,|^2-1+\bar{y}w).\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}\partial_y\log\Big(\frac{|\,1-\bar{w}y\,|}{|\,y-w\,|}\Big)&=\frac{1}{2}\frac{|\,y-w\,|^2}{|\,1-\bar{w}y\,|^2}\frac{(\overline{y-w})\,((1-\bar{y}w)\,|\,w\,|^2-1+\bar{y}w)}{|\,y-w\,|^4}\\&=\frac{1}{2}\frac{|\,w\,|^2-1}{(y-w)\,(1-\bar{w}y)}=\frac{1}{2}\Big(\frac{1}{w-y}-\frac{\bar{w}}{1-y\bar{w}}\Big).\qquad(6.13)\end{aligned}$$

Thus we achieve indeed for $y\in\bar{B}\setminus\{w\}$ and $w\in B$:

$$\begin{aligned}|\,\nabla_y\tilde{G}(w,y)\,|=2\;|\,\partial_y\tilde{G}(w,y)\,|&\le\frac{1}{2\pi}\Big(\frac{1}{|\,w-y\,|}+\frac{|\,\bar{w}\,|}{|\,1-y\bar{w}\,|}\Big)\\&\le\frac{1}{2\pi}\frac{1+|\,\bar{w}\,|}{|\,y-w\,|}<\frac{1}{\pi}\frac{1}{|\,y-w\,|},\end{aligned}$$

where we used $|\,\bar{w}\,|<1$ and $|\,w-y\,|\le|\,1-y\bar{w}\,|$ due to $|\,\kappa_w(y)\,|\le 1$ $\forall\, y\in\bar{B}\setminus\{w\}$. Hence, we gain immediately $\forall\, w\in B$:

$$\begin{aligned}\int_B|\,\nabla_y\tilde{G}(w,y)\,|^s\;dy<\frac{1}{\pi^s}\int_B\frac{1}{|\,y-w\,|^s}\,dy&\le 2\,\pi^{1-s}\int_0^2 r^{1-s}\,dr\\&=\pi^{1-s}\frac{2^{3-s}}{2-s},\end{aligned}$$

i.e. (6.6) and $\tilde{G}(w,\cdot\,)\in\mathring{H}^{1,s}(B)$, for $s<2$. Moreover also using (6.2) we achieve for any fixed $w\in B$:

$$\begin{aligned}\int_B|\,\nabla_w\tilde{G}(w,y)\,|^s\;dy=\int_B|\,\nabla_y\tilde{G}(y,w)\,|^s\;dy&<\frac{1}{\pi^s}\int_B\frac{1}{|\,w-y\,|^s}\,dy\\&\le\pi^{1-s}\frac{2^{3-s}}{2-s},\end{aligned}$$

thus (6.7). Furthermore we compute by (6.13):

$$\partial_{yy}\tilde{G}(w,y)=\frac{1}{4\pi}\Big(\frac{1}{(w-y)^2}-\frac{(\bar{w})^2}{(1-y\bar{w})^2}\Big).$$

Hence, again using $|\bar{w}|<1$ and $|y-w|\leq|1-y\bar{w}|\ \forall y\in\bar{B}\setminus\{w\}$ we arrive at the second asserted inequality in (6.8):

$$|\partial_{yy}\tilde{G}(w,y)|\leq\frac{1}{4\pi}\Big(\frac{1}{|w-y|^2}+\frac{|\bar{w}|^2}{|y-w|^2}\Big)<\frac{1}{2\pi}\frac{1}{|y-w|^2},$$

$\forall y\in\bar{B}\setminus\{w\}$. Moreover as κ_w satisfies Cauchy-Riemann's equations on $\bar{B}$ one easily computes that the Beltrami-Laplace operator $\triangle_{\kappa_w}$ w. r. to the transformation κ_w reduces to $\frac{1}{\det D_y\kappa_w}\triangle$ and we obtain indeed:

$$\triangle_y\tilde{G}(w,y)=\frac{1}{2\pi}\det D_y\kappa_w(y)\,\triangle_{\kappa_w}\Big(\log\frac{1}{|\cdot|}\circ\kappa_w(y)\Big)$$
$$=\frac{1}{2\pi}\det D_y\kappa_w(y)\,\triangle\Big(\log\frac{1}{|\cdot|}\Big)\circ\kappa_w(y)=0\qquad\forall y\in\bar{B}\setminus\{w\}.$$

Hence, we gain for the eigenvalues $\lambda^1_w(y)$ and $\lambda^2_w(y)$ of $D^2_y\tilde{G}(w,y)$ that $\lambda^1_w(y)=-\lambda^2_w(y)$, which implies that

$$\det D^2_y\tilde{G}(w,y)=\lambda^1_w(y)\lambda^2_w(y)=-\lambda^1_w(y)^2\leq 0\qquad\forall y\in\bar{B}\setminus\{w\}.\tag{6.14}$$

Furthermore denoting $y=y_1+iy_2$ we have

$$\begin{aligned}4\,\partial_{yy}\tilde{G}(w,y)&=(\partial_{y_1}-i\partial_{y_2})\circ(\partial_{y_1}-i\partial_{y_2})\tilde{G}(w,y)\\&=(\partial_{y_1y_1}-\partial_{y_2y_2}-i\,2\,\partial_{y_1y_2})\tilde{G}(w,y),\end{aligned}$$

which yields

$$\begin{aligned}&16\,|\partial_{yy}\tilde{G}(w,y)|^2\\&=((\partial_{y_1y_1}\tilde{G})^2-2\,\partial_{y_1y_1}\tilde{G}\,\partial_{y_2y_2}\tilde{G}+(\partial_{y_2y_2}\tilde{G})^2+4\,(\partial_{y_1y_2}\tilde{G})^2)(w,y)\\&=|D^2_y\tilde{G}(w,y)|^2-2\,\det D^2_y\tilde{G}(w,y),\end{aligned}$$

hence, together with (6.14) $|D^2_y\tilde{G}(w,y)|\leq 4\,|\partial_{yy}\tilde{G}(w,y)|\quad\forall y\in\bar{B}\setminus\{w\}$, which is the first asserted inequality in (6.8). Finally we shall derive formula (6.10) for $y\in\partial B$. To this end we recall by (6.3):

$$\begin{aligned}\partial_\nu\tilde{G}(w,y)&=\langle\nabla_y\tilde{G}(w,y),y\rangle=2\,\Re(\partial_yG(w,y)\,y)\\&=\frac{1}{2\pi}\Re\Big(\frac{y}{w-y}-\frac{\bar{w}y}{1-y\bar{w}}\Big).\end{aligned}\tag{6.15}$$

Moreover we have for $y\in\partial B$:

$$\frac{\bar{w}y}{1-y\bar{w}}=\frac{\bar{w}\bar{y}y}{\bar{y}-\bar{y}y\bar{w}}=\frac{\bar{w}}{\bar{y}-\bar{w}}.$$

Thus using $\Re\Big(\frac{\bar{w}}{\bar{y}-\bar{w}}\Big)=-\Re\Big(\frac{w}{w-y}\Big)$ we can conclude:

$$\begin{aligned}\Re\Big(\frac{y}{w-y}-\frac{\bar{w}y}{1-y\bar{w}}\Big)&=\Re\Big(\frac{y+w}{w-y}\Big)=\Re\Big(\frac{(y+w)\,(\overline{w-y})}{(w-y)\,(\overline{w-y})}\Big)\\&=\Re\Big(\frac{y\bar{w}-|y|^2+|w|^2-w\bar{y}}{|w-y|^2}\Big)=\frac{|w|^2-1}{|w-y|^2},\end{aligned}$$

$\forall\, y \in \partial B$, where we used $\Re(y\bar{w}) = \Re(w\bar{y})$. Thus together with (6.15) we achieve the asserted formula (6.10).

◇

In the sequel we will examine and use the Newtonian or Green's potential

$$\mathcal{G}(\varphi)(w) := \int_B \tilde{G}(w,y)\,\varphi(y)\,dy \qquad \textit{for } w \in \bar{B},$$

which is well defined for any $\varphi \in L^r(B)$, with $r > 1$, on account of $\tilde{G}(w,\cdot\,) \in L^p(B)$, $\forall\, p \in [1,\infty)$, $\forall\, w \in B$, and $\tilde{G}(w,\cdot\,) \equiv 0$ on B, $\forall\, w \in \partial B$, by the above proposition. Moreover we introduce the notation

$$C^2_0(B) := \{\varphi \in C^2(B) \cap C^0(\bar{B}) \mid \varphi\,|_{\partial B} \equiv 0\}. \tag{6.16}$$

Combining some of the above results with the residue calculus, p. 382 and p. 397 in [30], we derive now

Corollary 6.1 *For any $w \in B$ we have $\tilde{G}(w,\cdot\,) \in C^\infty(\bar{B} \setminus B_r(w))$, $\forall\, r < 1- \mid w \mid$. Furthermore there holds*

$$\int_{\partial B} \partial_\nu \tilde{G}(w,y)\,ds_y = -1 = \int_{\partial B_r(w)} \partial_\nu \tilde{G}(w,y)\,ds_y, \tag{6.17}$$

and for any $\varphi \in H^{2,2}(B) \cap C^2_0(B)$ and any $w \in B$:

$$-\varphi(w) = \int_B \tilde{G}(w,y)\,\triangle\varphi(y)\,dy \equiv \mathcal{G}(\triangle\varphi)(w). \tag{6.18}$$

Proof: For any fixed $w \in B$ there holds $\frac{|1-\bar{w}y|}{|w-y|} \in (0,\infty)$, for any $y \in \bar{B} \setminus B_r(w)$ and $r \in (0, 1- \mid w \mid)$, implying that $\tilde{G}(w,\cdot\,) \in C^\infty(\bar{B} \setminus B_r(w))$. Furthermore we gain by (6.10) $\forall\, w = (w_1,w_2) \in B$:

$$\int_{\partial B} \partial_\nu \tilde{G}(w,y)\,ds_y = \frac{\mid w \mid^2 - 1}{2\pi} \int_0^{2\pi} \frac{1}{\mid \cos t - w_1 \mid^2 + \mid \sin t - w_2 \mid^2}\,dt. \tag{6.19}$$

As in [30], p. 397, we associate to $R(y_1,y_2) := \frac{1}{|y_1-w_1|^2+|y_2-w_2|^2}$ the meromorphic function

$$\tilde{R}(z) := z^{-1} R\Big(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\Big)$$
$$= \frac{1}{(\frac{1}{2}(z^{\frac{3}{2}} + z^{-\frac{1}{2}}) - w_1\, z^{\frac{1}{2}})^2 + (\frac{1}{2i}(z^{\frac{3}{2}} - z^{-\frac{1}{2}}) - w_2\, z^{\frac{1}{2}})^2}$$

in order to use that

$$\int_0^{2\pi} \frac{1}{\mid \cos t - w_1 \mid^2 + \mid \sin t - w_2 \mid^2}\,dt = 2\pi \sum_{y \in B} res_y(\tilde{R}). \tag{6.20}$$

Now a simple computation yields for $w \in B \setminus \{0\}$ that w and $\frac{1}{\bar{w}}$ are the only zeroes of $\tilde{R}^{-1}$ in $\mathbb{C}$. Furthermore one can easily compute that

$$\partial_z(\tilde{R}^{-1})(z) = 1 - 2z\bar{w} + \mid w \mid^2 \qquad \forall\, z \in \mathbb{C} \setminus \{0\},$$

thus $\partial_z(\tilde{R}^{-1})(w) = 1 - \mid w \mid^2 \neq 0$. Hence, we conclude that w is a simple pole of $\tilde{R}$ and

$$res_w(\tilde{R}) = \frac{1}{\partial_z(\tilde{R}^{-1})(w)} = \frac{1}{1 - \mid w \mid^2}$$

by [30], p. 382. Combining this with (6.19) and (6.20) we achieve indeed the first equation in (6.17) for $w \in B \setminus \{0\}$. Finally for $w = 0$ one derives this identity immediately from (6.10). Now let $w \in B$ be arbitrarily fixed. Applying the divergence theorem to $\nabla_y \tilde{G}(w, \cdot\,) \in C^\infty(\bar{B} \setminus B_r(w))$ on the domain $B \setminus \overline{B_r(w)}$ and using (6.9) we obtain:

$$\int_{\partial B} \partial_\nu \tilde{G}(w,y)\, ds_y - \int_{\partial B_r(w)} \partial_\nu \tilde{G}(w,y)\, ds_y = \int_{B \setminus \overline{B_r(w)}} \triangle_y \tilde{G}(w,y)\, dy = 0,$$

i.e. the second equation in (6.17). Furthermore applying Green's second identity to $\tilde{G}(w, \cdot\,)$ and $\varphi \in C^2_0(B) \cap H^{2,2}(B)$ on the domain $B \setminus \overline{B_r(w)}$ we see again due to (6.9), $\tilde{G}(w, \cdot\,) \equiv 0$ on ∂B, $\varphi \equiv 0$ on ∂B and since we have $\partial_\nu \varphi \in L^2(\partial B)$ by the trace theorem for Sobolev functions on p. 249 in [1]:

$$\begin{aligned} & \int_{B \setminus \overline{B_r(w)}} \tilde{G}(w,y) \triangle \varphi(y)\, dy \qquad (6.21) \\ = & \int_{\partial B_r(w)} \partial_\nu \tilde{G}(w,y)\, \varphi(y)\, ds_y - \int_{\partial B_r(w)} \tilde{G}(w,y)\, \partial_\nu \varphi(y)\, ds_y. \end{aligned}$$

We note that for $y \in \partial B_r(w)$ there holds:

$$\frac{r}{2} < \mid \kappa_w(y) \mid = \frac{\mid y - w \mid}{\mid \bar{w}y - 1 \mid} < \frac{r}{1 - \mid w \mid}. \qquad (6.22)$$

Hence, we obtain for $r < 1 - \mid w \mid$:

$$\mid \int_{\partial B_r(w)} \tilde{G}(w,y)\, \partial_\nu \varphi(y)\, ds_y \mid < r \log \frac{2}{r} \max_{\partial B_r(w)} \mid \nabla \varphi \mid \longrightarrow 0, \qquad (6.23)$$

for $r \searrow 0$, due to $\varphi \in C^2(B)$. Furthermore using (6.5), (6.17) and that $\mathrm{osc}_{\overline{B_r(w)}}(\varphi) \longrightarrow 0$, for $r \searrow 0$, one achieves

$$
\begin{aligned}
&\Big| \int_{\partial B_r(w)} \partial_\nu \tilde{G}(w,y)\,\varphi(y)\,ds_y + \varphi(w) \Big| \\
&= \Big| \int_{\partial B_r(w)} \partial_\nu \tilde{G}(w,y)\,(\varphi(y)-\varphi(w))\,ds_y \Big| \\
&\le \int_{\partial B_r(w)} |\nabla_y \tilde{G}(w,y)|\,|\varphi(y)-\varphi(w)|\;ds_y \\
&\le \int_{\partial B_r(w)} \frac{1}{\pi}\frac{1}{|y-w|}\,ds_y\; osc_{\overline{B_r(w)}}(\varphi) \\
&= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{r}\,r\,d\theta\; osc_{\overline{B_r(w)}}(\varphi) = 2\,osc_{\overline{B_r(w)}}(\varphi) \longrightarrow 0 \qquad \text{for } r \searrow 0.
\end{aligned}
$$

Combining this with (6.21), (6.23) and $\triangle\varphi, \tilde{G}(w,\cdot\,) \in L^2(B)$ we gain (6.18) in the limit for $r \searrow 0$.

$\diamond$

Now we are able to prove

Proposition 6.2 *The weak limit $G(\cdot\,,y)$ in (5.9) coincides with the classical Green function $\tilde{G}(\cdot\,,y)$ in (6.1) on $\bar{B}$, for any $y \in B$.*

Proof: We fix some $y \in B$ arbitrarily. From (5.2) and (5.9) we derive the existence of some null sequence $\{\rho_j\}$ such that there holds $\forall\,\varphi \in C_c^\infty(B)$:

$$
\begin{array}{ccc}
\int_B \nabla_w G^{\rho_j}(w,y)\cdot\nabla\varphi(w)\,dw & = & \frac{1}{|B_{\rho_j}(y)|}\int_{B_{\rho_j}(y)} \varphi(w)\,dw \\
\downarrow & & \downarrow \qquad\qquad \star \\
\int_B \nabla_w G(w,y)\cdot\nabla\varphi(w)\,dw & = & \varphi(y),
\end{array}
$$

for $j \to \infty$, thus especially

$$\int_B \nabla_w G(w,y)\cdot\nabla\varphi(w)\,dw = 0 \qquad \forall\,\varphi \in C_c^\infty(B \setminus \overline{B_R(y)}), \tag{6.24}$$

for any fixed $R \in (0, \frac{1-|y|}{2})$. As we already know $G(\cdot\,,y) \in H^{1,2}(B \setminus \overline{B_R(y)})$, for $R \in (0, \frac{1-|y|}{2})$, by (5.16) we can reformulate (6.24) as

$$\triangle G(\cdot\,,y) \equiv 0 \qquad \text{weakly on } B \setminus \overline{B_R(y)}. \tag{6.25}$$

Since we also know that $G(\cdot\,,y) \equiv 0$ on ∂B by (5.9) we achieve from the L^2-theory, Theorem 8.13 in [8], that $G(\cdot\,,y) \in C^\infty(\bar{B} \setminus B_r(y))$, $\forall\,r \in (R, 1-|\,y\,|)$. Furthermore since we have $G(\cdot\,,y) \in \mathring{H}^{1,s}(B)$, $\forall\,s \in (1,2)$, by

(5.9) we may apply the divergence theorem to $G(\,\cdot\,,y)\nabla\varphi$ for any $\varphi \in C^\infty_c(B)$ which yields due to $\star$:

$$\varphi(y) = \int_B \nabla_w G(w,y)\cdot\nabla\varphi(w)\,dw = -\int_B G(w,y)\,\triangle\varphi(w)\,dw. \tag{6.26}$$

Now by the symmetry of $\tilde{G}(\,\cdot\,,\,\cdot\,)$ in y and w on $(\bar{B}\times\bar{B})\setminus\Lambda$ we derive from (6.18):

$$\varphi(w) = -\int_B \tilde{G}(y,w)\,\triangle\varphi(y)\,dy,$$

for any fixed $w\in B$. Hence, exchanging the names of w and y in the above formula we conclude that there holds for any fixed $y \in B$:

$$\varphi(y) = -\int_B \tilde{G}(w,y)\,\triangle\varphi(w)\,dw.$$

Hence, subtracting this from (6.26) we achieve:

$$0 = \int_B (\tilde{G}(w,y) - G(w,y))\,\triangle\varphi(w)\,dw, \tag{6.27}$$

$\forall\,\varphi\in C^\infty_c(B)$. By (5.10) and $\tilde{G}(\,\cdot\,,y)\in L^p(B)$, $\forall\,p\in[1,\infty)$, due to Prop. 6.1 and especially the symmetry of $\tilde{G}(\,\cdot\,,\,\cdot\,)$ in y and w we conclude that $\tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y) \in L^p(B)$ for any $p\in[1,\infty)$. Hence, (6.27) implies by Weyl's lemma that $\tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y)$ is harmonic on B and especially of class $C^\infty(B)$. Now together with $\tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y) \in C^\infty(\bar{B}\setminus B_r(y))$, $\forall\,r\in(R,1-\mid y\mid)$, we see that $\tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y)\in C^\infty(\bar{B})$. Therefore we may apply the maximum principle for harmonic functions to $\tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y)$ yielding indeed:

$$\max_{\bar{B}} \mid \tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y)\mid = \max_{\partial B}\mid \tilde{G}(\,\cdot\,,y) - G(\,\cdot\,,y)\mid = 0,$$

where we used that $G(\,\cdot\,,y)\equiv 0\equiv\tilde{G}(\,\cdot\,,y)$ on ∂B by (5.9) and Prop. 6.1.

$\diamond$

Hence, we may write G for $\tilde{G}$ in the sequel. Now we shall combine some results of Prop. 6.1 with standard methods of [8], Lemma 4.1, 4.2, for the investigation of the Newtonian potential in order to prove

Theorem 6.1 *For any $\varphi\in C^2_0(B)\cap C^{0,\alpha}(\bar{B})$, for some $\alpha\in(0,1)$, (see (6.16)) there holds $\mathcal{G}(\varphi)\in C^2(B)\cap \mathring{H}^{1,r}(B)$, $\forall\,r\in[1,\infty)$, with*

$$\frac{\partial}{\partial w_i}\mathcal{G}(\varphi)(w) = \int_B \frac{\partial}{\partial w_i} G(w,y)\,\varphi(y)\,dy \qquad \text{and} \tag{6.28}$$

$$\begin{aligned}\frac{\partial}{\partial w_j \partial w_i}\mathcal{G}(\varphi)(w) = &\int_B \frac{\partial}{\partial w_j\partial w_i} G(w,y)\,(\varphi(y)-\varphi(w))\,dy \\ &+\varphi(w)\int_{\partial B}\frac{\partial}{\partial w_i}G(w,y)\,\nu_j\,ds_y\end{aligned} \tag{6.29}$$

$\forall\, w \in B$ and $i,j \in \{1,2\}$, where ν denotes the outward unit normal about ∂B. This yields especially

$$\triangle \mathcal{G}(\varphi) = -\varphi \qquad \text{on } B. \tag{6.30}$$

Proof: Firstly we see that the three integrals in (6.28) and (6.29) exist due to the estimates (6.5) and (6.8) together with the mean value theorem and $\varphi \in C^{0,\alpha}(\bar{B})$, $\alpha \in (0,1)$. Now as in Lemma 4.1 in [8] we choose some function $\eta \in C^\infty(\mathbb{R})$ with $0 \le \eta \le 1$, $\eta(t) \equiv 0$ for $t \le 1$ and $\eta(t) \equiv 1$ for $t \ge 2$, $0 \le \eta' \le 2$, $\mid \eta'' \mid \le$ const., construct the symmetric cut-off functions $\eta_\epsilon(w,y) := \eta\big(\frac{|w-y|}{\epsilon}\big)$, for $\epsilon > 0$ and $(w,y) \in \bar{B} \times \bar{B}$, and introduce the family

$$\mathcal{G}(\varphi)_\epsilon(w) := \int_B G(w,y)\, \eta_\epsilon(w,y)\, \varphi(y)\, dy$$

for $w \in B$ and $\epsilon > 0$. We see immediately that $\{\mathcal{G}(\varphi)_\epsilon\} \subset C^1(B)$ and that

$$\mathcal{G}(\varphi)_\epsilon \longrightarrow \mathcal{G}(\varphi) \qquad in\ C^0(\bar{B}'), \tag{6.31}$$

for $\epsilon \searrow 0$ and any domain $B' \subset\subset B$, by $\varphi \in L^\infty(B)$, (5.11) and Lebesgue's convergence theorem, thus especially $\mathcal{G}(\varphi) \in C^0(B)$. Now we fix some $w \in B$ arbitrarily and obtain for $2\epsilon < 1- \mid w \mid$ and some $i \in \{1,2\}$:

$$\begin{aligned} &\Big| \int_B \frac{\partial}{\partial w_i} G(w,y)\, \varphi(y)\, dy - \frac{\partial}{\partial w_i} \mathcal{G}(\varphi)_\epsilon(w) \Big| \qquad (6.32)\\ &= \Big| \int_{B_{2\epsilon}(w)} \frac{\partial}{\partial w_i} \big((1-\eta_\epsilon(w,y))\, G(w,y)\big)\, \varphi(y)\, dy \Big| \\ &\le \| \varphi \|_{L^\infty(B)} \Big(\int_{B_{2\epsilon}(w)} \Big| \frac{\partial}{\partial w_i} G(w,y) \Big|\, dy + \frac{2}{\epsilon} \int_{T_{2\epsilon}(w)} G(w,y)\, dy \Big), \end{aligned}$$

where we set $T_{2\epsilon}(w) := B_{2\epsilon}(w) \setminus \overline{B_\epsilon(w)}$. For the first integral we obtain immediately by (6.5):

$$\begin{aligned} \int_{B_{2\epsilon}(w)} \Big| \frac{\partial}{\partial w_i} G(w,y) \Big|\, dy &\le \frac{1}{\pi} \int_{B_{2\epsilon}(w)} \frac{1}{\mid w-y \mid}\, dy \\ &= 2 \int_0^{2\epsilon} \frac{1}{r}\, r\, dr = 4\epsilon \longrightarrow 0, \end{aligned} \tag{6.33}$$

for $\epsilon \searrow 0$, uniformly for any $w \in \bar{B}' \subset\subset B$. In order to estimate the second integral we derive as in (6.22) for $y \in T_{2\epsilon}(w)$:

$$\frac{\epsilon}{2} < \mid \kappa_w(y) \mid = \frac{\mid y - w \mid}{\mid \bar{w}y - 1 \mid} < \frac{2\epsilon}{1- \mid w \mid}.$$

Hence, together with (6.11) we can compute by the transformation theorem, for $2\epsilon < 1- \mid w \mid$:

$$
\begin{aligned}
&\frac{2}{\epsilon} \int_{T_{2\epsilon}(w)} G(w,y)\,dy \\
&< \frac{16}{(1- \mid w \mid^2)^2 \pi\,\epsilon} \int_{T_{2\epsilon}(w)} \Big(\log \frac{1}{\mid \cdot \mid} \Big) \circ \kappa_w(y) \, \det D_y\kappa_w(y)\,dy \\
&= \frac{16}{(1- \mid w \mid^2)^2 \pi\,\epsilon} \int_{\kappa_w(T_{2\epsilon}(w))} \log \frac{1}{\mid z \mid}\,dz \\
&< \frac{16}{(1- \mid w \mid^2)^2 \pi\,\epsilon} \int_{B_{\frac{2\epsilon}{1-|w|}}(0) \setminus \overline{B_{\frac{\epsilon}{2}}(0)}} \log \frac{1}{\mid z \mid}\,dz \\
&< \frac{16}{(1- \mid w \mid^2)^2 \pi\,\epsilon} \mid B_{\frac{2\epsilon}{1-|w|}}(0) \setminus \overline{B_{\frac{\epsilon}{2}}(0)} \mid \log \frac{2}{\epsilon} \\
&< \frac{64\,\epsilon}{(1- \mid w \mid^2)^2 (1- \mid w \mid)^2} \log \frac{2}{\epsilon} \longrightarrow 0 \qquad \textit{for } \epsilon \searrow 0,
\end{aligned}
$$

uniformly for any $w \in \bar{B}' \subset\subset B$. Hence, we achieve together with (6.32) and (6.33) that

$$
\frac{\partial}{\partial w_i} \mathcal{G}(\varphi)_\epsilon \longrightarrow \int_B \frac{\partial}{\partial w_i} G(\,\cdot\,,y)\,\varphi(y)\,dy \qquad \textit{in } C^0(\bar{B}'), \tag{6.34}
$$

for any domain $B' \subset\subset B$. Thus together with (6.31) we finally achieve

$$
\mathcal{G}(\varphi)_\epsilon \longrightarrow \mathcal{G}(\varphi) \qquad \textit{in } C^1_{loc}(B), \tag{6.35}
$$

for $\epsilon \searrow 0$, and therefore in particular $\mathcal{G}(\varphi) \in C^1(B)$ and the first equation in (6.28) by (6.34), $\forall\, w \in B$. Furthermore we have $\mathcal{G}(\varphi) \equiv 0$ on ∂B due to $G(\,\cdot\,,y) \equiv 0$ on ∂B, for any $y \in B$, by Prop. 6.1 and 6.2. Now combining the symmetry of $G(\,\cdot\,,\,\cdot\,)$ on $(B \times B) \setminus \Lambda$ with (5.11), for $p = 2$, and Hölder's inequality we obtain for any $r \in [1, \infty)$:

$$
\begin{aligned}
&\int_B \mid \mathcal{G}(\varphi)(w) \mid^r \; dw = \int_B \mid \int_B G(w,y)\,\varphi(y)\,dy \mid^r \; dw \\
&\leq \int_B \Big(\Big(\int_B \mid G(w,y) \mid^2 \; dy \Big)^{\frac{1}{2}} \Big(\int_B \mid \varphi(y) \mid^2 \; dy \Big)^{\frac{1}{2}} \Big)^r dw \\
&\leq \textit{const.}(2)^r \, \pi \parallel \varphi \parallel^r_{L^2(B)} < \infty,
\end{aligned}
$$

which shows firstly that $\mathcal{G}(\varphi) \in L^r(B)$, for any $r \in [1, \infty)$. Next applying (6.28), Hölder's inequality and (6.7) we compute for any $r \in [1, \infty)$:

$$
\begin{aligned}
&\int_B \mid \nabla \mathcal{G}(\varphi)(w) \mid^r \; dw = \int_B \mid \int_B \nabla_w G(w,y)\,\varphi(y)\,dy \mid^r \; dw \\
&\leq \int_B \Big(\Big(\int_B \mid \nabla_w G(w,y) \mid^{\frac{3}{2}} \; dy \Big)^{\frac{2}{3}} \Big(\int_B \mid \varphi(y) \mid^3 \; dy \Big)^{\frac{1}{3}} \Big)^r dw \\
&\leq \textit{const.} \Big(\frac{3}{2} \Big)^r \pi \parallel \varphi \parallel^r_{L^3(B)} < \infty.
\end{aligned}
$$

Thus together with $\mathcal{G}(\varphi) \in C^1(B)$ we conclude that $\mathcal{G}(\varphi)$ is of class $H^{1,r}(B)$, $\forall\, r \in [1,\infty)$, and in particular of class $C^0(\bar{B})$. Hence, taking also $\mathcal{G}(\varphi) \equiv 0$ on ∂B into account we finally infer that $\mathcal{G}(\varphi) \in \mathring{H}^{1,r}(B)$, $\forall\, r \in [1,\infty)$. Now making use of the estimates (6.5) and (6.8) and the formula (6.28) for the first derivatives of $\mathcal{G}(\varphi)$ we shall follow the lines of the proof of Lemma 4.2 in [8] in order to prove that $\mathcal{G}(\varphi) \in C^2(B)$ together with the formula in (6.29) for its second derivatives. To this end we define

$$u(w) := \int_B \frac{\partial^2}{\partial w_j \partial w_i} G(w,y)\,(\varphi(y)-\varphi(w))\,dy + \varphi(w) \int_{\partial B} \frac{\partial}{\partial w_i} G(w,y)\,\nu_j\,ds_y,$$

for some arbitrarily chosen $i,j \in \{1,2\}$ and

$$v_\epsilon(w) := \frac{\partial}{\partial w_i} \mathcal{G}_\epsilon(\varphi)(w) = \int_B \frac{\partial}{\partial w_i} (G(w,y)\,\eta_\epsilon(w,y))\,\varphi(y)\,dy,$$

for $2\epsilon < 1- \mid w \mid$. We note that $v_\epsilon \in C^1(B)$ and derive by the symmetry of $G(w,y)\,\eta_\epsilon(w,y)$ in y and w on $(\bar{B}\times\bar{B}) \setminus \Lambda$ as in (6.2):

$$\frac{\partial^s}{\partial y^s}(G(w^*,y^*)\,\eta_\epsilon(w^*,y^*)) = \frac{\partial^s}{\partial w^s}(G(y^*,w^*)\,\eta_\epsilon(y^*,w^*)), \tag{6.36}$$

for any fixed $w^* \in B$ and $y^* \in \bar{B} \setminus \{w^*\}$ and any $s = (s_1,s_2) \in \{0,1,2\}^2$ with $\mid s \mid := s_1 + s_2 \le 2$, and therefore together with the divergence theorem applied to $\frac{\partial}{\partial y_i}(G(\,\cdot\,,w)\,\eta_\epsilon(\,\cdot\,,w)) \in C^\infty(\bar{B})$:

$$\begin{aligned} \frac{\partial}{\partial w_j} v_\epsilon(w) &= \int_B \frac{\partial^2}{\partial w_j \partial w_i}(G(w,y)\,\eta_\epsilon(w,y))\,\varphi(y)\,dy \\ &= \int_B \frac{\partial^2}{\partial y_j \partial y_i}(G(y,w)\,\eta_\epsilon(y,w))\,\varphi(y)\,dy \\ = \int_B \frac{\partial^2}{\partial y_j \partial y_i}&(G(y,w)\,\eta_\epsilon(y,w))\,(\varphi(y)-\varphi(w))\,dy \\ &+\varphi(w) \int_B \frac{\partial^2}{\partial y_j \partial y_i}(G(y,w)\,\eta_\epsilon(y,w))\,dy \\ = \int_B \frac{\partial^2}{\partial y_j \partial y_i}&(G(y,w)\,\eta_\epsilon(y,w))\,(\varphi(y)-\varphi(w))\,dy \\ &+\varphi(w) \int_{\partial B} \frac{\partial}{\partial w_i} G(w,y)\,\nu_j\,ds_y, \end{aligned}$$

for $2\epsilon < 1- \mid w \mid$, where we used (6.2) and $\eta_\epsilon(\,\cdot\,, w) \equiv 1$ on $B \setminus B_{2\epsilon}(w)$ in the last line. Now by this property of η_ϵ we compute:

$$\begin{aligned}
\mid u(w) - \frac{\partial}{\partial w_j} v_\epsilon(w) \mid =\mid \int_B \frac{\partial^2}{\partial y_j \partial y_i} (G(y,w)\,(1-\eta_\epsilon(y,w))\,(\varphi(y)-\varphi(w))\,dy \mid \\
=\mid \int_{B_{2\epsilon}(w)} \Big(\frac{\partial}{\partial y_j}\Big(\frac{\partial}{\partial y_i} G(y,w)\,(1-\eta_\epsilon(y,w))\Big) \\
-\frac{\partial}{\partial y_j} G(y,w)\,\frac{\partial}{\partial y_i}\eta_\epsilon(y,w) - G(y,w)\,\frac{\partial^2}{\partial y_j \partial y_i}\eta_\epsilon(y,w)\Big)\,(\varphi(y)-\varphi(w))\,dy \mid \\
\leq \| \nabla\varphi \|_{L^\infty(B_{2\epsilon}(w))} \int_{B_{2\epsilon}(w)} \Big(\mid \frac{\partial}{\partial y_j}\Big(\frac{\partial}{\partial y_i} G(y,w)\,(1-\eta_\epsilon(y,w))\Big) \mid \\
+ \mid \frac{\partial}{\partial y_j} G(y,w)\,\frac{\partial}{\partial y_i}\eta_\epsilon(y,w) \mid + G(y,w)\,\mid \frac{\partial^2}{\partial y_j \partial y_i}\eta_\epsilon(y,w) \mid \Big) \mid y-w \mid \,dy \\
=: \| \nabla\varphi \|_{L^\infty(B_{2\epsilon}(w))}\, (I(\epsilon) + II(\epsilon) + III(\epsilon)), \quad (6.37)
\end{aligned}$$

for $2\epsilon < 1- \mid w \mid$, where we used $\varphi \in C^2(B)$. Firstly we can estimate:

$$\begin{aligned}
I(\epsilon) \leq \int_{B_{2\epsilon}(w)} \Big(\frac{2}{\epsilon}\frac{1}{2\pi}\frac{1}{\mid y-w \mid} + \frac{1}{2\pi}\frac{1}{\mid y-w \mid^2}\Big) \mid y-w \mid \,dy \\
= \int_0^{2\epsilon} \frac{2}{\epsilon} r + 1\,dr = 6\epsilon. \qquad (6.38)
\end{aligned}$$

Next we estimate:

$$II(\epsilon) \leq \int_{B_{2\epsilon}(w)} \frac{2}{\epsilon}\frac{1}{2\pi}\frac{1}{\mid y-w \mid} \mid y-w \mid \,dy = \frac{2}{\epsilon}\int_0^{2\epsilon} r\,dr = 4\epsilon, \qquad (6.39)$$

for $2\epsilon < 1- \mid w \mid$. Finally we shall examine the most challenging term $III(\epsilon)$. To this end we compute:

$$\begin{aligned}
\frac{\partial}{\partial y_i}\eta_\epsilon(y,w) = \eta'\Big(\frac{\mid y-w \mid}{\epsilon}\Big)\frac{1}{\epsilon}\frac{y_i - w_i}{\mid y-w \mid}, \\
\frac{\partial^2}{\partial y_j \partial y_i}\eta_\epsilon(y,w) = \eta''\Big(\frac{\mid y-w \mid}{\epsilon}\Big)\frac{1}{\epsilon^2}\frac{(y_j-w_j)(y_i-w_i)}{\mid y-w \mid^2} \\
+\eta'\Big(\frac{\mid y-w \mid}{\epsilon}\Big)\frac{1}{\epsilon}\frac{\mid y-w \mid \delta_{ij} - (y_i-w_i)\,\frac{y_j-w_j}{\mid y-w \mid}}{\mid y-w \mid^2},
\end{aligned}$$

thus

$$\mid \frac{\partial^2}{\partial y_j \partial y_i}\eta_\epsilon(y,w) \mid \leq \frac{const.(\eta)}{\epsilon^2} + \frac{4}{\epsilon}\frac{1}{\mid y-w \mid},$$

$\forall\,(y,w)\in(\bar{B}\times\bar{B})\setminus\Lambda$ and any $i,j\in\{1,2\}$. Thus we can estimate by Hölder's inequality and (5.11):

$$
\begin{aligned}
III(\epsilon) \le \Big(\int_{B_{2\epsilon}(w)} G(y,w)^3\,dy\Big)^{\frac{1}{3}} \Big(\int_{B_{2\epsilon}(w)} \Big(\mid \frac{\partial^2}{\partial y_j \partial y_i}\eta_\epsilon(y,w) \mid\mid y-w\mid\Big)^{\frac{3}{2}}\,dy\Big)^{\frac{2}{3}} \\
\le const.(3)\,(2\pi)^{\frac{2}{3}}\Big(\int_0^{2\epsilon}\Big(\Big(\frac{const.(\eta)}{\epsilon^2}+\frac{4}{\epsilon}\frac{1}{r}\Big)\,r\Big)^{\frac{3}{2}}\,r\,dr\Big)^{\frac{2}{3}} \\
\le const.(3)\,(2\pi)^{\frac{2}{3}}\Big(\int_0^{2\epsilon}\sqrt{2}\,\Big(\frac{const.(\eta)^{\frac{3}{2}}}{\epsilon^3}\,r^{\frac{5}{2}}+\Big(\frac{4}{\epsilon}\Big)^{\frac{3}{2}} r\Big)\,dr\Big)^{\frac{2}{3}} \\
= const.(3)\,(2\pi)^{\frac{2}{3}}\Big(\int_0^{2\epsilon}\sqrt{2}\,\Big(\frac{const.(\eta)^{\frac{3}{2}}}{\epsilon^3}\,r^{\frac{5}{2}}+\Big(\frac{4}{\epsilon}\Big)^{\frac{3}{2}} r\Big)\,dr\Big)^{\frac{2}{3}} \\
= const.(3)\,(2\pi)^{\frac{2}{3}}\Big(\frac{2^5}{7}\,const.(\eta)^{\frac{3}{2}}+2^{\frac{9}{2}}\Big)^{\frac{2}{3}}\epsilon^{\frac{1}{3}}, \qquad (6.40)
\end{aligned}
$$

where we used the inequality

$$(a+b)^{\frac{3}{2}} \le \sqrt{2}\,(a^{\frac{3}{2}}+b^{\frac{3}{2}}),$$

for any $a,b\ge 0$, which follows immediately from the convexity of the function $f(x):=x^{\frac{3}{2}}$ for $x\ge 0$. Hence, inserting (6.38), (6.39) and (6.40) into (6.37) we arrive at

$$
\mid u(w)-\frac{\partial}{\partial w_j}v_\epsilon(w)\mid \le \parallel \nabla\varphi \parallel_{L^\infty(B_{2\epsilon}(w))}\ (I(\epsilon)+II(\epsilon)+III(\epsilon)) \qquad (6.41)
$$

$$
\le \parallel \nabla\varphi \parallel_{L^\infty(B_{2\epsilon}(w))}\ \Big(6\epsilon+4\epsilon+const.(3)\,(2\pi)^{\frac{2}{3}}\Big(\frac{2^5}{7}\,const.(\eta)^{\frac{3}{2}}+2^{\frac{9}{2}}\Big)^{\frac{2}{3}}\epsilon^{\frac{1}{3}}\Big) \longrightarrow 0,
$$

for $\epsilon\searrow 0$, uniformly for any $w\in\bar{B}'$ and any subdomain $B'\subset\subset B$. Therefore we achieve by $\frac{\partial}{\partial w_j}v_\epsilon=\frac{\partial}{\partial w_j\partial w_i}\mathcal{G}_\epsilon(\varphi)$ by definition of v_ϵ and the $C^1_{loc}(B)$-convergence of $\{\mathcal{G}_\epsilon(\varphi)\}$ in (6.35) that

$$\mathcal{G}_\epsilon(\varphi)\longrightarrow\mathcal{G}(\varphi) \qquad in\ C^2_{loc}(B),$$

which implies $\mathcal{G}(\varphi)\in C^2(B)$ and moreover in combination with (6.41):

$$u(w)=\lim_{\epsilon\searrow 0}\frac{\partial}{\partial w_j}v_\epsilon(w)=\lim_{\epsilon\searrow 0}\frac{\partial}{\partial w_j\partial w_i}\mathcal{G}_\epsilon(\varphi)(w)=\frac{\partial^2}{\partial w_j\partial w_i}\mathcal{G}(\varphi)(w),$$

for any $w\in B$, just as asserted in (6.29). Now combining this with (6.9) and (6.17) we obtain $\forall\,w\in B$:

$$
\begin{aligned}
\triangle\mathcal{G}(\varphi)(w) &= \sum_{i=1}^{2}\varphi(w)\int_{\partial B}\frac{\partial}{\partial w_i}G(w,y)\,\nu_i\,ds_y \\
&= \varphi(w)\int_{\partial B}\frac{\partial}{\partial\nu}G(w,y)\,ds_y = -\varphi(w).
\end{aligned}
$$

$\diamond$

Together with the L^2-regularity theory, Theorem 8.12 in [8], we can prove the Calderon-Zygmund inequality now also for our potential $\mathcal{G}$ in the special case $p = 2$ and equation (6.30) even for any $\varphi \in L^2(B)$:

Theorem 6.2 *For any $\varphi \in L^2(B)$ there holds the estimate*

$$\| \mathcal{G}(\varphi) \|_{H^{2,2}(B)} \leq const. \| \varphi \|_{L^2(B)}, \tag{6.42}$$

$\mathcal{G}(\varphi) \in \mathring{H}^{1,2}(B) \cap H^{2,2}(B)$ and also equation (6.30) in $L^2(B)$ for any $\varphi \in L^2(B)$.

Proof: The above theorem guarantees that for any $\varphi \in C_c^\infty(B)$ there holds $\mathcal{G}(\varphi) \in C^2(B)$, thus $\nabla\mathcal{G}(\varphi)\,\eta \in C_c^1(B)\ \forall\,\eta \in C_c^\infty(B)$. Therefore we derive from (6.30) due to the divergence theorem:

$$\int_B \nabla\mathcal{G}(\varphi) \cdot \nabla\eta \, dw = \int_B \varphi\, \eta \, dw, \tag{6.43}$$

$\forall\,\eta \in C_c^\infty(B)$, and as we also know by the above theorem that $\nabla\mathcal{G}(\varphi) \in L^2(B)$ we achieve (6.43) even $\forall\,\eta \in \mathring{H}^{1,2}(B)$ by approximation, i.e. (6.30) weakly on B. Now using that $\mathcal{G}(\varphi) \in \mathring{H}^{1,2}(B)$ the L^2-regularity theory, Theorem 8.12 in [8], yields that $\mathcal{G}(\varphi)$ lies in $H^{2,2}(B)$, satisfying the estimate

$$\| \mathcal{G}(\varphi) \|_{H^{2,2}(B)} \leq const.(\| \mathcal{G}(\varphi) \|_{L^2(B)} + \| \varphi \|_{L^2(B)}) \leq const. \| \varphi \|_{L^2(B)}, \tag{6.44}$$

where we also used the continuity of $\mathcal{G} : L^2(B) \longrightarrow L^\infty(B)$ which can be derived from estimate (5.11) together with the symmetry of $G(\,\cdot\,,\,\cdot\,) \equiv \tilde{G}(\,\cdot\,,\,\cdot\,)$:

$$| \mathcal{G}(\varphi)(w) | \leq \| G(w,\,\cdot\,) \|_{L^2(B)} \| \varphi \|_{L^2(B)} \leq const.(2) \| \varphi \|_{L^2(B)}, \tag{6.45}$$

$\forall\, w \in B$. Now let some $\varphi \in L^2(B)$ be given arbitrarily. Then we choose some sequence $\{\varphi_j\} \subset C_c^\infty(B)$ satisfying $\varphi_j \longrightarrow \varphi$ in $L^2(B)$. Thus by (6.45) we also know that

$$\mathcal{G}(\varphi_j) \longrightarrow \mathcal{G}(\varphi) \qquad \text{in } L^2(B) \tag{6.46}$$

and by (6.44) that $\| \mathcal{G}(\varphi_j) \|_{H^{2,2}(B)} \leq const.$, implying the existence of a subsequence $\{\varphi_{j_k}\}$ which satisfies

$$\mathcal{G}(\varphi_{j_k}) \rightharpoonup \mathcal{G}(\varphi) \qquad \text{weakly in } H^{2,2}(B).$$

Hence by the weak lower semicontinuity of the norm $\| \cdot \|_{H^{2,2}(B)}$ we obtain together with (6.44) indeed:

$$\| \mathcal{G}(\varphi) \|_{H^{2,2}(B)} \leq \liminf_{k\to\infty} \| \mathcal{G}(\varphi_{j_k}) \|_{H^{2,2}(B)} \leq const. \liminf_{k\to\infty} \| \varphi_{j_k} \|_{L^2(B)}$$
$$= const. \| \varphi \|_{L^2(B)},$$

for any $\varphi \in L^2(B)$. Now using this estimate and the linearity of our potential $\mathcal{G}$ we obtain:

$$\| \mathcal{G}(\varphi_j) - \mathcal{G}(\varphi) \|_{H^{2,2}(B)} = \| \mathcal{G}(\varphi_j - \varphi) \|_{H^{2,2}(B)} \leq \mathit{const.} \| \varphi_j - \varphi \|_{L^2(B)} \longrightarrow 0.$$

Now together with the closedness of $\mathring{H}^{1,2}(B) \cap H^{2,2}(B)$ in $H^{2,2}(B)$ w. r. to the $H^{2,2}(B)$–norm we achieve that $\mathcal{G}(\varphi) \in \mathring{H}^{1,2}(B) \cap H^{2,2}(B)$ and also the equation (6.30) in $L^2(B)$ for any $\varphi \in L^2(B)$ immediately.

$\diamond$

Now defining the domain of the minimal Laplace operator to be $\mathrm{Dom}(\dot{\triangle}) := H^{2,2}(B) \cap C^2_0(B)$ Theorems 6.1 and 6.2 imply in particular

Corollary 6.2 image($\dot{\triangle}$) *is densely contained in* $L^2(B)$.

Proof: Let some $\varphi \in L^2(B)$ be arbitrarily chosen and some sequence $\{\varphi_j\} \subset C^\infty_c(B)$ with $\varphi_j \longrightarrow \varphi$ in $L^2(B)$. Now by Theorems 6.1 and 6.2 there holds $\mathcal{G}(\varphi_j) \in H^{2,2}(B) \cap C^2_0(B) = \mathrm{Dom}(\dot{\triangle})$ and $\dot{\triangle}(-\mathcal{G}(\varphi_j)) = \varphi_j$, $\forall j \in \mathbb{N}$, hence

$$\dot{\triangle}(-\mathcal{G}(\varphi_j)) = \varphi_j \longrightarrow \varphi \qquad \text{in } L^2(B).$$

$\diamond$

Chapter 7

The Schwarz operators A^τ and $\dot{A}^\tau$ for $\tau \in K(\tilde{f})$

7.1 $\boldsymbol{Dom}\big(\overline{\dot{A}^\tau}\big) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ for $\tau \in K(\tilde{f})$

We consider the Schwarz operator $A^\tau := -\triangle + 2\,(KE)^\tau$, for $\tau \in K(\tilde{f})$, on the domain

$$Dom(A^\tau) := \{\varphi \in C^2(B) \cap \mathring{H}^{1,2}(B) \mid A^\tau(\varphi) \in L^2(B)\} \tag{7.1}$$

and the minimal Schwarz operator $\dot{A}^\tau$ on the domain $H^{2,2}(B) \cap C_0^2(B)$. Using estimate (2.38) we are able to prove assertion (3.11) in [18]:

Proposition 7.1 *For any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and any $\tau \in K(\tilde{f})$ there holds*

$$\mid (\mathrm{KE})^\tau \varphi(w) \mid \leq c(\tau, \alpha) \sum_{k=1}^{N+3} \mid w - e^{i\tau_k} \mid^{-1+\frac{\alpha}{2}} \| \dot{\triangle}\varphi \|_{L^2(B)} \qquad \forall\, w \in B. \tag{7.2}$$

Proof: Let $\tau \in K(\tilde{f})$ and $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ be arbitrarily chosen. If we combine Green's identity (6.18) with $G(\,\cdot\,,\,\cdot\,) \geq 0$ and Hölder's inequality we obtain for any $p_1 \in (0,1)$, any $p_2 \in (1,2)$ and $p_3 := \big(1 - \frac{p_2}{2}\big)^{-1} \in (2, \infty)$:

$$\begin{aligned} \mid \varphi(w) \mid &= \mid \int_B G(w,y)^{1-p_1}\, G(w,y)^{p_1}\, \dot{\triangle}\varphi(y)\, dy \mid \\ &\leq \Big(\int_B (G(w,y)^{1-p_1})^{p_2'}\, dy \Big)^{\frac{1}{p_2'}} \Big(\int_B G(w,y)^{p_1 p_2} \mid \dot{\triangle}\varphi(y) \mid^{p_2}\, dy \Big)^{\frac{1}{p_2}} \qquad (7.3) \\ &\leq const.(p_1, p_2') \Big(\int_B G(w,y)^{p_1 p_2 p_3}\, dy \Big)^{\frac{1}{p_2 p_3}} \| \dot{\triangle}\varphi \|_{L^2(B)}, \end{aligned}$$

where we applied (5.11) for $p := (1-p_1)p_2'$ (see (7.5) below), for $p := (1-p_1)p_2'$ together with the symmetry of $\tilde{G}(\,\cdot\,,\,\cdot\,) \equiv G(\,\cdot\,,\,\cdot\,)$ and that we have by $p_3^{-1} = 1 - \frac{p_2}{2}$:

$$p_2 p_3' = p_2\Big(1 - \frac{1}{p_3}\Big)^{-1} = p_2\Big(1 - 1 + \frac{p_2}{2}\Big)^{-1} = 2.$$

Now combining (6.5) with the symmetry of $\tilde{G}(\,\cdot\,,\,\cdot\,) \equiv G(\,\cdot\,,\,\cdot\,)$ and thus the fact that $G(\,\cdot\,,y) \equiv 0$ on ∂B, for any fixed $y \in B$, by Prop. 6.1, and with (6.2), we infer from the mean value theorem for an arbitrarily chosen $k \in \{1,\dots,N+3\}$ the existence of some point ξ_k on the open segment $(w, e^{i\tau_k})$ such that

$$\begin{aligned} G(w,y) &= \mid G(w,y) - G(e^{i\tau_k},y) \mid \leq \mid \nabla_w G(\xi_k,y) \mid \mid w - e^{i\tau_k} \mid \\ &= \mid \nabla_y G(y,\xi_k) \mid \mid w - e^{i\tau_k} \mid \leq \frac{1}{\pi}\,\frac{1}{\mid \xi_k - y \mid} \mid w - e^{i\tau_k} \mid, \end{aligned} \tag{7.4}$$

for any $y \in B \setminus [w, e^{i\tau_k})$. Now we choose $p_1 := 1 - \frac{\alpha}{2}$ and $p_2 := 1 + \frac{\alpha}{4}$ and compute:

$$(1-p_1)\,p_2' = \frac{\alpha}{2}\Big(1 - \frac{1}{\frac{\alpha}{4}}\Big)^{-1} = \frac{\alpha}{2}\Big(\frac{1+\frac{\alpha}{4}-1}{1+\frac{\alpha}{4}}\Big)^{-1} = 2 + \frac{\alpha}{2} \in (1,\infty), \tag{7.5}$$

$$p_1 p_2 p_3 = \Big(1 - \frac{\alpha}{2}\Big)\frac{1+\frac{\alpha}{4}}{1-\frac{1}{2}-\frac{\alpha}{8}} = \frac{1-\frac{\alpha}{4}-\frac{\alpha^2}{8}}{\frac{1}{2}-\frac{\alpha}{8}} \in (1,2).$$

Hence, combining this with (7.3) and (7.4) we can estimate:

$$\begin{aligned} &\mid \varphi(w) \mid \leq const.(p_1,p_2')\Big(\int_{B\setminus[w,e^{i\tau_k})} G(w,y)^{p_1p_2p_3}\,dy\Big)^{\frac{1}{p_2p_3}} \parallel \dot{\triangle}\varphi \parallel_{L^2(B)} \\ &\leq const.(\alpha)\Big(\int_B \frac{1}{\mid \xi_k - y \mid^{p_1p_2p_3}}\,dy\Big)^{\frac{1}{p_2p_3}} \mid w - e^{i\tau_k} \mid^{1-\frac{\alpha}{2}} \parallel \dot{\triangle}\varphi \parallel_{L^2(B)} \\ &\qquad\qquad \leq c(\alpha) \mid w - e^{i\tau_k} \mid^{1-\frac{\alpha}{2}} \parallel \dot{\triangle}\varphi \parallel_{L^2(B)}, \end{aligned} \tag{7.6}$$

for each $k \in \{1,\dots,N+3\}$. Hence, together with estimate (2.38) and the existence of $(KE)^\tau$ in any point of B we conclude:

$$\begin{aligned} \mid (KE)^\tau \varphi(w) \mid &\leq const.(\tau)\sum_{k=1}^{N+3} \mid w - e^{i\tau_k} \mid^{-2+\alpha} c(\alpha) \mid w - e^{i\tau_k} \mid^{1-\frac{\alpha}{2}} \parallel \dot{\triangle}\varphi \parallel_{L^2(B)} \\ &\leq c(\tau,\alpha)\sum_{k=1}^{N+3} \mid w - e^{i\tau_k} \mid^{-1+\frac{\alpha}{2}} \parallel \dot{\triangle}\varphi \parallel_{L^2(B)} \qquad \forall\, w \in B. \end{aligned}$$

◇

In order to extend the above estimate onto $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ we have to prove

Proposition 7.2 *$H^{2,2}(B) \cap C_0^2(B)$ is densely contained in $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ w. r. to the $H^{2,2}(B)-$norm.*

Proof: Let $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ be chosen arbitrarily. We shall follow the proof of Satz 2.23 on p. 108 in [1]. Firstly we consider the open annuli

$$U_k := \{w \in B \mid 2^{-k} < dist(w, \partial B) < 4\, 2^{-k}\},$$

for $k \in \mathbb{N}$, which yield a covering of B, i.e. $\bigcup_{k=1}^{\infty} U_k = B$, and a subordinate smooth partition of unity $\{\eta_k\}$. One can easily see that any fixed point $w^* \in B$ is contained in exacty two overlapping sets $U_{k_1(w^*)}$, $U_{k_2(w^*)}$, $k_2(w^*) = k_1(w^*) + 1$, i.e. with $\eta_k(w^*) = 0$ if and only if $k \neq k_1(w^*), k_2(w^*)$. Now we define

$$\mathcal{S}_\delta(\varphi) := \int_B K_\delta(\,\cdot\, - y)\, \varphi(y)\, dy \quad \in C^\infty(\mathbb{R}^2),$$

with some symmetric Dirac family $\{K_\delta\}$ with supp $K_\delta = \overline{B_\delta(0)}$ and $\int_{B_\delta(0)} K_\delta\, dy \equiv 1$, $\forall\, \delta > 0$. Now we fix some $\epsilon > 0$ arbitrarily. As we know by [1], p. 108, that

$$\mathcal{S}_\delta(\varphi) \longrightarrow \varphi \qquad in\ \ H^{2,2}_{loc}(B),$$

for $\delta \searrow 0$, we can assign to each k some sufficiently small real number $Z(k) \in (0, \frac{1}{2^k\,\epsilon})$ such that $\{Z(k)\}$ is monotonically decreasing and such that the functions

$$\varphi_{k,\epsilon} := \mathcal{S}_{Z(k)\epsilon}(\varphi) \quad \in C^\infty(\mathbb{R}^2)$$

satisfy

$$\parallel \varphi_{k,\epsilon} - \varphi \parallel_{H^{2,2}(U_k)} < c_k\, \epsilon, \tag{7.7}$$

where we set

$$c_k := 2^{-k}(\parallel \eta_k \parallel_{C^2(\bar{B})} + 1)^{-1}, \tag{7.8}$$

for each $k \in \mathbb{N}$. Now we define the function

$$\varphi_\epsilon := \sum_{k \in \mathbb{N}} \eta_k\, \varphi_{k,\epsilon} \qquad on\ \ \bar{B}$$

and note that $\varphi_\epsilon \in C^\infty(B)$. As in the proof of Satz 2.23 on p. 108 in [1] one can see by (7.7), (7.8) and supp $\eta_k = \bar{U}_k$ that

$$\begin{aligned} \parallel D^s\varphi_\epsilon - D^s\varphi \parallel_{L^2(B)} &\leq const. \sum_{k\in\mathbb{N}} \parallel \eta_k \parallel_{C^2(\bar{B})} \parallel \varphi_{k,\epsilon} - \varphi \parallel_{H^{s,2}(U_k)} \\ &\leq const. \sum_{k\in\mathbb{N}} \parallel \eta_k \parallel_{C^2(\bar{B})} c_k\, \epsilon < const.\, \epsilon \sum_{k\in\mathbb{N}} 2^{-k} = const. \epsilon, \end{aligned}$$

for $s = 0, 1, 2$ and any fixed $\epsilon > 0$, thus indeed

$$\varphi_\delta \longrightarrow \varphi \qquad in\ \ H^{2,2}(B), \tag{7.9}$$

for $\delta \searrow 0$, which shows $\varphi_\epsilon \in H^{2,2}(B)$ in particular. Now we choose some $w^* \in B$ arbitrarily and abbreviate $k_j := k_j(w^*)$, for $j = 1, 2$. Then we estimate by the properties of $\{K_\delta\}$:

$$| \varphi_{k_j,\epsilon}(w^*) | \leq \int_B K_{Z(k_j)\epsilon}(w^* - y) \, | \varphi(y) | \, dy$$
$$= \int_{B_{Z(k_j)\epsilon}(w^*)} K_{Z(k_j)\epsilon}(w^* - y) \, | \varphi(y) | \, dy \leq \| \varphi \|_{L^\infty(B_{Z(k_1)\epsilon}(w^*))} \qquad (7.10)$$

for $j = 1, 2$, where we used $Z(k_1) > Z(k_2)$ and

$$Z(k_1)\epsilon < 2^{-k_1} \leq dist(\bar{U}_{k_1}, \partial B) < dist(w^*, \partial B),$$

implying $B_{Z(k_1)\epsilon}(w^*) \subset\subset B$. Moreover we compute by $w^* \in U_{k_1}$:

$$\sup\{dist(w, \partial B) \mid w \in B_{Z(k_1)\epsilon}(w^*)\} = dist(w^*, \partial B) + Z(k_1)\epsilon$$
$$< 4\,2^{-k_1} + 2^{-k_1} = 5\,2^{-k_1}. \qquad (7.11)$$

Now let $\bar{w} \in \partial B$ be arbitrarily fixed and $\{w_i\} \subset B$ some sequence with $w_i \longrightarrow \bar{w}$. Applying (7.11) to each point w_i we conclude:

$$\sup\{dist(w, \partial B) \mid w \in B_{Z(k_1(w_i))\epsilon}(w_i)\} < 5\,2^{-k_1(w_i)} \qquad \forall\, i \in \mathbb{N},$$

which implies by $\varphi \in H^{2,2}(B) \subset C^{0,\alpha}(\bar{B})$, $\forall\, \alpha \in (0,1)$, $\varphi \equiv 0$ on ∂B and $k_1(w_i) \to \infty$:

$$\| \varphi \|_{L^\infty(B_{Z(k_1(w_i))\epsilon}(w_i))} \leq const.(\alpha)\, 5^\alpha\, 2^{-k_1(w_i)\,\alpha} \longrightarrow 0, \qquad (7.12)$$

for $i \to \infty$. Hence, we can estimate by (7.10) and (7.12):

$$| \varphi_\epsilon(w_i) | \leq \sum_{k \in \mathbb{N}} \eta_k(w_i) \, | \varphi_{k,\epsilon}(w_i) | = \sum_{j=1,2} \eta_{k_j(w_i)}(w_i) \, | \varphi_{k_j(w_i),\epsilon}(w_i) |$$
$$\leq 2 \, \| \varphi \|_{L^\infty(B_{Z(k_1(w_i))\epsilon}(w_i))} \longrightarrow 0 \qquad \textit{for } i \to \infty. \qquad (7.13)$$

On the other hand we know $\varphi_\epsilon \in H^{2,2}(B) \subset C^0(\bar{B})$, which implies

$$\varphi_\epsilon(w_i) \longrightarrow \varphi_\epsilon(\bar{w}) \qquad \textit{for } w_i \to \bar{w},$$

and therefore $\varphi_\epsilon(\bar{w}) = 0$ due to (7.13), for any $\bar{w} \in \partial B$. Hence, we proved $\varphi_\epsilon \,|_{\partial B} \equiv 0$. As we also have $\varphi_\epsilon \in C^\infty(B)$ we can conclude that $\varphi_\epsilon \in H^{2,2}(B) \cap C_0^2(B)$ in particular, for any $\epsilon > 0$, which together with (7.9) proves the assertion of the proposition.

$\diamond$

Thus we obtain

Corollary 7.1 *Estimate (7.2) extends onto* $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ *for any* $\tau \in K(\tilde{f})$.

Proof: This follows immediately from Prop. 7.2 together with the continuity of the Sobolev embedding $H^{2,2}(B) \hookrightarrow C^0(\bar{B})$.

$\diamond$

Now we are able to prove

Proposition 7.3 *For any* $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ *and any* $\tau \in K(\tilde{f})$ *there holds:*

$$\| \, 2(KE)^\tau \, \varphi \, \|_{L^2(B)} \leq \frac{1}{2} \, \| \, \triangle \varphi \, \|_{L^2(B)} + c \, \| \, \varphi \, \|_{L^2(B)}, \tag{7.14}$$

for some constant $c = c(\tau)$ *that only depends on* τ.

Proof: We fix some $\tau \in K(\tilde{f})$. If this statement was incorrect, then for each $m \in \mathbb{N}$ there would have to exist some $\varphi_m \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ such that

$$\frac{1}{2} \, \| \, \triangle \varphi_m \, \|_{L^2(B)} + m \, \| \, \varphi_m \, \|_{L^2(B)} < \| \, 2(KE)^\tau \, \varphi_m \, \|_{L^2(B)} \, .$$

Thus for the normalized functions $\psi_m := \frac{\varphi_m}{\| 2(KE)^\tau \varphi_m \|_{L^2(B)}} \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ we could conclude:

$$\frac{1}{2} \, \| \, \triangle \psi_m \, \|_{L^2(B)} + m \, \| \, \psi_m \, \|_{L^2(B)} < 1 \qquad \forall \, m \in \mathbb{N}. \tag{7.15}$$

Now from this we infer firstly:

$$\| \, \triangle \psi_m \, \|_{L^2(B)} < 2 - 2m \, \| \, \psi_m \, \|_{L^2(B)} \leq const. \qquad \forall \, m \in \mathbb{N}.$$

Hence, the right hand side of (7.2) yields an $L^2(B)$-majorant for the sequence $\{(KE)^\tau \, \psi_m\}$. Secondly we derive from (7.15):

$$\| \, \psi_m \, \|_{L^2(B)} < \frac{1}{m} \Big(1 - \frac{1}{2} \, \| \, \triangle \psi_m \, \|_{L^2(B)} \Big) \longrightarrow 0,$$

for $m \to \infty$. Thus we obtain a subsequence $\{\psi_{m_k}\}$ satisfying

$$(KE)^\tau \psi_{m_k}(w) \longrightarrow 0 \qquad \text{for a.e. } w \in B,$$

for $k \to \infty$, where we recall that $(KE)^\tau$ exists in almost every point of B. Hence, Lebesgue's convergence theorem implies:

$$\| \, 2(KE)^\tau \, \psi_{m_k} \, \|_{L^2(B)} \longrightarrow 0,$$

which contradicts the normalization $\| \, 2(KE)^\tau \, \psi_m \, \|_{L^2(B)} = 1 \;\; \forall \, m \in \mathbb{N}$.

$\diamond$

From this we can derive firstly that $Dom(\dot{A}^{\tau}) = H^{2,2}(B) \cap C_0^2(B)$ is contained in $Dom(A^{\tau})$, thus $\dot{A}^{\tau} \subset A^{\tau}$, and especially that A^{τ} is densely defined in $L^2(B)$, $\forall\, \tau \in K(\tilde{f})$. Moreover we have

Proposition 7.4 *A^{τ} is symmetric w. r. to $\langle \cdot\, , \cdot \rangle_{L^2(B)}$, i.e. $A^{\tau} \subset (A^{\tau})^*$, $\forall\, \tau \in K(\tilde{f})$.*

Proof: We fix some $\tau \in K(\tilde{f})$. For any $\varphi \in Dom(A^{\tau})$ and $\psi \in C_c^{\infty}(B)$ we have $\nabla\varphi\, \psi \in C_c^1(B)$. Hence, by the divergence theorem we obtain:

$$\langle A^{\tau}(\varphi), \psi \rangle_{L^2(B)} = \int_B \nabla\varphi \cdot \nabla\psi + 2\,(KE)^{\tau} \varphi\psi \, dw =: \mathcal{L}^{\tau}(\varphi, \psi). \tag{7.16}$$

Now let $\psi \in \mathring{H}^{1,2}(B)$ be arbitrarily chosen and $\{\psi_j\} \subset C_c^{\infty}(B)$ with $\psi_j \longrightarrow \psi$ in $\mathring{H}^{1,2}(B)$. By Hölder's inequality and Sobolev's embedding theorem we achieve due to $1 - \frac{2}{2} = 0 > 0 - \frac{2}{q}$, $\forall\, q \in [1, \infty)$:

$$\begin{aligned} \|\, (KE)^{\tau}\, \varphi\, (\psi_j - \psi)\, \|_{L^1(B)} &\leq \|\, (KE)^{\tau}\, \|_{L^{p^*}(B)} \|\, \varphi\, \|_{L^r(B)} \|\, \psi_j - \psi\, \|_{L^q(B)} \\ &\leq \|\, (KE)^{\tau}\, \|_{L^{p*}(B)} \|\, \varphi\, \|_{L^r(B)}\ const.(q)\ \|\, \psi_j - \psi\, \|_{H^{1,2}(B)} \longrightarrow 0, \end{aligned}$$

for $j \to \infty$, with $\frac{1}{p^*} + \frac{1}{r} + \frac{1}{q} = 1$ and $p^* \in (1, \frac{2}{2-\alpha})$. Hence, recalling that $A^{\tau}(\varphi) \in L^2(B)$ we gain (7.16) in the limit also for $\psi \in \mathring{H}^{1,2}(B)$, thus especially for any $\psi \in Dom(A^{\tau})$. Together with the symmetry of $\mathcal{L}^{\tau}(\,\cdot\, , \cdot\,)$ this yields for an arbitrary $\varphi \in Dom(A^{\tau})$:

$$\langle A^{\tau}(\varphi), \psi \rangle_{L^2(B)} = \mathcal{L}^{\tau}(\varphi, \psi) = \mathcal{L}^{\tau}(\psi, \varphi) = \langle \varphi, A^{\tau}(\psi) \rangle_{L^2(B)}, \tag{7.17}$$

$\forall\, \psi \in Dom(A^{\tau})$, which shows indeed $Dom(A^{\tau}) \subset Dom((A^{\tau})^*)$ and $(A^{\tau})^*(\varphi) = A^{\tau}(\varphi)$, just as asserted.

$\diamond$

From this we derive

Corollary 7.2 *A^{τ}, $\dot{A}^{\tau}$ and $\dot{\triangle}$ are closable in $L^2(B)$, $\forall\, \tau \in K(\tilde{f})$.*

Proof: We fix some $\tau \in K(\tilde{f})$ and show the assertion for A^{τ}. Let $\{\varphi_j\} \subset Dom(A^{\tau})$ be a sequence satisfying

$$\varphi_j \longrightarrow 0 \qquad \text{and} \qquad A^{\tau}(\varphi_j) \longrightarrow v \qquad \text{in } L^2(B),$$

then we infer from Prop. 7.4 especially for any $\psi \in C_c^{\infty}(B)$:

$$\begin{array}{ccc} \langle A^{\tau}(\varphi_j), \psi \rangle_{L^2(B)} & = & \langle \varphi_j, A^{\tau}(\psi) \rangle_{L^2(B)} \\ \downarrow & & \downarrow \\ \langle v, \psi \rangle_{L^2(B)} & = & \langle 0, A^{\tau}(\psi) \rangle_{L^2(B)} = 0. \end{array}$$

Hence, we conclude that $v \equiv 0$, which was to be shown. For $\dot{A}^{\tau}$ and $\dot{\triangle}$ the assertion also follows from their symmetry on $H^{2,2}(B) \cap C_0^2(B) \subset Dom(A^{\tau})$.

$\diamond$

Now we can prove

Corollary 7.3 *There holds* $\mathrm{Dom}(\bar{\dot{\triangle}}) = \mathrm{Dom}(\overline{\dot{A}^\tau}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\forall\, \tau \in K(\tilde{f})$.

Proof: We fix some $\tau \in K(\tilde{f})$ arbitrarily and choose some $\varphi \in Dom(\bar{\dot{\triangle}})$. Thus there is a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B) = Dom(\dot{\triangle})$ such that

$$\varphi_m \longrightarrow \varphi \qquad \text{and} \qquad \dot{\triangle}\varphi_m \longrightarrow \bar{\dot{\triangle}}(\varphi) \qquad \text{in} \quad L^2(B). \tag{7.18}$$

By (7.14) we see that

$$\| \, 2(KE)^\tau(\varphi_n - \varphi_m) \, \|_{L^2(B)} \leq \frac{1}{2} \, \| \, \dot{\triangle}\varphi_n - \dot{\triangle}\varphi_m \, \|_{L^2(B)} + c \, \| \, \varphi_n - \varphi_m \, \|_{L^2(B)}, \tag{7.19}$$

thus that $\{2(KE)^\tau\varphi_m\}$ is a Cauchy sequence in $L^2(B)$. Now from (7.18) we can deduce the pointwise convergence

$$(KE)^\tau\varphi_{m_k}(w) \longrightarrow (KE)^\tau\varphi(w) \qquad \text{for a.e. } w \in B, \tag{7.20}$$

for some suitable sequence $\{m_k\}$, which shows that $(KE)^\tau\,\varphi_m \longrightarrow (KE)^\tau\,\varphi$ in $L^2(B)$ and therefore again with (7.18):

$$\dot{A}^\tau(\varphi_m) = -\dot{\triangle}\varphi_m + 2(KE)^\tau\varphi_m \longrightarrow -\bar{\dot{\triangle}}(\varphi) + 2(KE)^\tau\varphi = \overline{\dot{A}^\tau}(\varphi)$$

in $L^2(B)$, which proves that $\varphi \in Dom(\overline{\dot{A}^\tau})$.

Now let some $\varphi \in Dom(\overline{\dot{A}^\tau})$ be given arbitrarily, which means that there exists a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B)$ satisfying

$$\varphi_m \longrightarrow \varphi \qquad \text{and} \qquad \dot{A}^\tau(\varphi_m) \longrightarrow \overline{\dot{A}^\tau}(\varphi) \qquad \text{in} \quad L^2(B). \tag{7.21}$$

For some arbitrary $\psi \in H^{2,2}(B) \cap C_0^2(B)$ we have by (7.14):

$$\begin{aligned} \| \, \dot{A}^\tau(\psi) \, \|_{L^2(B)} &\geq \| \, \dot{\triangle}\psi \, \|_{L^2(B)} - \| \, 2(KE)^\tau\psi \, \|_{L^2(B)} \\ &\geq \frac{1}{2} \, \| \, \dot{\triangle}\psi \, \|_{L^2(B)} - c \, \| \, \psi \, \|_{L^2(B)}, \end{aligned}$$

and therefore

$$\| \, \dot{\triangle}\psi \, \|_{L^2(B)} \leq 2 \, \| \, \dot{A}^\tau(\psi) \, \|_{L^2(B)} + 2c \, \| \, \psi \, \|_{L^2(B)} \, .$$

Combining this with (7.21) we conclude that $\{\dot{\triangle}\varphi_m\}$ is a Cauchy sequence in $L^2(B)$, and therefore also $\{2\,(KE)^\tau\varphi_m\} = \{\dot{\triangle}\varphi_m + \dot{A}^\tau(\varphi_m)\}$ due to the second convergence in (7.21). Now due to the first convergence in (7.21) we conclude again (7.20) and thus $(KE)^\tau\,\varphi_m \longrightarrow (KE)^\tau\,\varphi$ in $L^2(B)$ and therefore again with the second convergence in (7.21):

$$\dot{\triangle}\varphi_m = -\dot{A}^\tau(\varphi_m) + 2(KE)^\tau\varphi_m \longrightarrow -\overline{\dot{A}^\tau}(\varphi) + 2(KE)^\tau\varphi = \bar{\dot{\triangle}}\varphi$$

in $L^2(B)$, i.e. that $\varphi \in Dom(\bar{\dot{\triangle}})$.
Finally we have to prove that $Dom(\bar{\dot{\triangle}}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$. Firstly let $\varphi \in Dom(\bar{\dot{\triangle}})$ be chosen arbitrarily, thus there exists a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B) = Dom(\dot{\triangle})$ satisfying (7.18). By (6.18), the Calderon-Zygmund inequality (6.42) for our potential $\mathcal{G}$ and (7.18) we achieve:

$$\| \varphi_m \|_{H^{2,2}(B)} = \| \mathcal{G}(\dot{\triangle}\varphi_m) \|_{H^{2,2}(B)} \leq \textit{const.} \| \dot{\triangle}\varphi_m \|_{L^2(B)} \leq \textit{const.}, \quad (7.22)$$

$\forall\, m \in \mathbb{N}$. Hence, together with the compactness of the embedding $H^{2,2}(B) \hookrightarrow L^2(B)$ and (7.18) we achieve the existence of a subsequence $\{\varphi_{m_k}\}$ such that

$$\varphi_{m_k} \rightharpoonup \varphi \qquad \text{weakly in } H^{2,2}(B). \quad (7.23)$$

This shows indeed $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ as $\mathring{H}^{1,2}(B)$ is closed in particular w. r. to weak $H^{2,2}(B)-$convergence and $\mathring{H}^{1,2}(B) \supset Dom(\dot{\triangle})$. Finally the inclusion $H^{2,2}(B) \cap \mathring{H}^{1,2}(B) \subset Dom(\bar{\dot{\triangle}})$ follows immediately from the approximation result of Prop. 7.2.

$\diamond$

7.2 Essential self-adjointness of $\dot{A}^\tau$ and A^τ

Firstly we prove the essential self-adjointness of $\dot{\triangle}$, which requires

Proposition 7.5 *There is a constant $a > 0$ such that*

$$\| \dot{\triangle}\varphi \|_{L^2(B)} \geq a \| \varphi \|_{L^2(B)} \qquad \forall\, \varphi \in H^{2,2}(B) \cap C_0^2(B). \quad (7.24)$$

Proof: If this statement was wrong then there would exist a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C_0^2(B)$ such that

$$\| \dot{\triangle}\varphi_m \|_{L^2(B)} < \frac{1}{m} \| \varphi_m \|_{L^2(B)} \qquad \forall\, m \in \mathbb{N}.$$

Hence, the normalized functions $\psi_m := \frac{\varphi_m}{\|\varphi_m\|_{L^2(B)}} \in H^{2,2}(B) \cap C_0^2(B)$ would satisfy:

$$\| \dot{\triangle}\psi_m \|_{L^2(B)} < \frac{1}{m} \longrightarrow 0. \quad (7.25)$$

Since $\mathcal{G} : L^2(B) \longrightarrow L^2(B)$ is continuous by Theorem 6.2 we could derive from (6.18) and (7.25) that

$$\psi_m = -\mathcal{G}(\dot{\triangle}\psi_m) \longrightarrow 0 \qquad \text{in } L^2(B),$$

which contradicts $\| \psi_m \|_{L^2(B)} = 1 \ \forall\, m \in \mathbb{N}$.

$\diamond$

Now we can prove as in [37], p. 59:

Theorem 7.1 *$\dot{\triangle}$ is essentially self-adjoint w. r. to $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(B)}$, i.e. $\bar{\dot{\triangle}} = (\bar{\dot{\triangle}})^*$.*

Proof: Firstly one can easily see that $\bar{\dot{\triangle}} \subset \big(\bar{\dot{\triangle}}\big)^*$. For let $\varphi, \psi \in Dom(\bar{\dot{\triangle}})$ be arbitrarily chosen, then there are sequences $\{\varphi_m\}$, $\{\psi_m\} \subset H^{2,2}(B) \cap C^2_0(B)$ that satisfy (7.18) respectively. Hence, Green's second formula, which holds true for pairs of functions in $H^{2,2}(B) \cap C^2_0(B)$, yields in the limit:

$$\begin{array}{c} \langle \dot{\triangle}\varphi_m, \psi_m\rangle_{L^2(B)} = \langle \varphi_m, \dot{\triangle}\psi_m\rangle_{L^2(B)} \\ \downarrow \qquad\qquad\qquad \downarrow \\ \\ \langle \bar{\dot{\triangle}}\varphi, \psi\rangle_{L^2(B)} = \langle \varphi, \bar{\dot{\triangle}}\psi\rangle_{L^2(B)}, \end{array}$$

which shows that $\varphi \in Dom\big((\bar{\dot{\triangle}})^*\big)$ with $\big(\bar{\dot{\triangle}}\big)^*(\varphi) = \bar{\dot{\triangle}}(\varphi)$. Now we prove the opposite inclusion $\big(\bar{\dot{\triangle}}\big)^* \subset \bar{\dot{\triangle}}$. To this end we firstly recall that image$(\dot{\triangle})$ is densely contained in $L^2(B)$ by Corollary 6.2. Hence, for any $g \in Dom\big((\bar{\dot{\triangle}})^*\big)$ there exists a sequence $\{\varphi_m\} \subset H^{2,2}(B) \cap C^2_0(B) = Dom(\dot{\triangle})$ such that

$$\dot{\triangle}\varphi_m \longrightarrow \big(\bar{\dot{\triangle}}\big)^*(g) \qquad \text{in } L^2(B). \tag{7.26}$$

This implies in particular that $\{\dot{\triangle}\varphi_m\}$ is a Cauchy sequence in $L^2(B)$, hence, $\{\varphi_m\}$ turns out to be a Cauchy sequence in $L^2(B)$ as well on account of estimate (7.24). Thus there exists some $\varphi \in L^2(B)$ such that $\varphi_m \longrightarrow \varphi$ in $L^2(B)$. Now combining this with the symmetry of $\dot{\triangle}$, i.e. with Green's second formula, and with (7.26) we achieve for any $\psi \in Dom(\dot{\triangle})$:

$$\begin{aligned} \langle g - \varphi, \dot{\triangle}\psi\rangle_{L^2(B)} &= \langle g, \bar{\dot{\triangle}}\psi\rangle_{L^2(B)} - \langle \varphi, \dot{\triangle}\psi\rangle_{L^2(B)} \\ &= \langle \big(\bar{\dot{\triangle}}\big)^*(g), \psi\rangle_{L^2(B)} - \lim_{m\to\infty} \langle \varphi_m, \dot{\triangle}\psi\rangle_{L^2(B)} \\ &= \langle \big(\bar{\dot{\triangle}}\big)^*(g), \psi\rangle_{L^2(B)} - \lim_{m\to\infty} \langle \dot{\triangle}\varphi_m, \psi\rangle_{L^2(B)} = 0. \end{aligned}$$

As we know that image$(\dot{\triangle})$ is densely contained in $L^2(B)$ by Corollary 6.2 this implies that $g = \varphi$, thus $\varphi_m \longrightarrow g$ in $L^2(B)$. Hence, together with (7.26) we can conclude that $\bar{\dot{\triangle}}(g) \equiv L^2(B) - \lim(\{\dot{\triangle}\varphi_m\}) = \big(\bar{\dot{\triangle}}\big)^*(g)$ and $g \in Dom\big(\bar{\dot{\triangle}}\big)$, which completes the proof.

$\diamond$

Together with estimate (7.14), for $\tau \in K(\tilde{f})$, and the obvious symmetry of $(KE)^\tau$ we infer from Theorem 4.4 in [24], p. 288:

Corollary 7.4 $\dot{A}^\tau = -\dot{\triangle} + 2\,(KE)^\tau$ *is essentially self-adjoint w. r. to* $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(B)}$, *i.e.* $\overline{\dot{A}^\tau} = \big(\overline{\dot{A}^\tau}\big)^*$, $\forall\,\tau \in K(\tilde{f})$.

◇

Now combining Prop. 7.4 with the fact that $Dom(A^\tau)$ is densely contained in $L^2(B)$ w. r. to $\|\cdot\|_{L^2(B)}$ we can derive by twice application of Theorem 5.29 in [24], p. 168:

Corollary 7.5 $(A^\tau)^*$ *is densely defined in* $L^2(B)$ *and closed,* $(A^\tau)^{**} = \bar{A}^\tau$ *and* $(A^\tau)^* = \overline{(A^\tau)^*} = ((A^\tau)^*)^{**}$, $\forall\,\tau \in K(\tilde{f})$.

◇

Summarizing all our results we achieve:

Theorem 7.2 $\big(\dot{A}^\tau\big)^* = \overline{\dot{A}^\tau} = \bar{A}^\tau = (A^\tau)^*$ *are self-adjoint operators with domain* $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\forall\,\tau \in K(\tilde{f})$.

Proof: We fix some $\tau \in K(\tilde{f})$. Firstly there holds by Prop. 7.4: $\dot{A}^\tau \subset A^\tau \subset (A^\tau)^*$. Combining this with Corollaries 7.4 and 7.5 we achieve:

$$\big(\overline{\dot{A}^\tau}\big)^* = \overline{\dot{A}^\tau} \subset \bar{A}^\tau \subset \overline{(A^\tau)^*} = ((A^\tau)^*)^{**} = ((A^\tau)^{**})^* = (\bar{A}^\tau)^* \subset \big(\overline{\dot{A}^\tau}\big)^*.$$

Hence, also noting that $\overline{(A^\tau)^*} = (A^\tau)^*$ by Corollary 7.5, we can conclude:

$$\overline{\dot{A}^\tau} = \bar{A}^\tau = (A^\tau)^*$$

are self-adjoint operators with domain $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ by Corollary 7.4. Furthermore applying Theorem 5.29 in [24], p. 168, to the densely defined and closable operator $\dot{A}^\tau$ we obtain that $\big(\dot{A}^\tau\big)^*$ is densely defined in $L^2(B)$, closed, i.e. $\big(\dot{A}^\tau\big)^* = \overline{\big(\dot{A}^\tau\big)^*}$, and $\big(\dot{A}^\tau\big)^{**} = \overline{\dot{A}^\tau}$. Now applying it to the densely defined and closed operator $\big(\dot{A}^\tau\big)^*$ again we gain that $\big((\dot{A}^\tau)^*\big)^{**} = \overline{\big(\dot{A}^\tau\big)^*}$. Hence, we achieve together with Corollary 7.4 that

$$\overline{\dot{A}^\tau} = \big(\overline{\dot{A}^\tau}\big)^* = \big((\dot{A}^\tau)^{**}\big)^* = \big((\dot{A}^\tau)^*\big)^{**} = \overline{\big(\dot{A}^\tau\big)^*} = \big(\dot{A}^\tau\big)^*.$$

◇

Chapter 8

The spectra and eigenspaces of $A^{\mathcal{T}}$ and $\bar{A}^{\mathcal{T}}$

We will denote $S\mathring{H}^{1,2}(B) := \{\varphi \in \mathring{H}^{1,2}(B) \,|\, \|\varphi\|_{L^2(B)} = 1\}$, and analogously $S(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))$ and $SDom(A^\tau)$. As in (7.16) we will use the bilinear form

$$\mathcal{L}^\tau(\varphi, \psi) := \int_B \nabla\varphi \cdot \nabla\psi + 2\,(KE)^\tau \varphi\psi\, dw,$$

for φ, $\psi \in \mathring{H}^{1,2}(B)$, thus especially $J^\tau(\varphi) \equiv \mathcal{L}^\tau(\varphi, \varphi)$. We fix some $\tau \in K(\tilde{f})$ and $p^* \in (1, \frac{2}{2-\alpha})$ arbitrarily and abbreviate $A := A^\tau$, $\mathcal{L} := \mathcal{L}^\tau$ and $J := J^\tau$. Firstly we need

Proposition 8.1 *There exists some constant $C(p^*)$ such that:*

$$J(\varphi) \geq \frac{1}{2} \int_B |\nabla\varphi|^2 \; dw - C(p^*) \,\|\, KE \,\|_{L^{p^*}(B)} \qquad \forall\, \varphi \in S\mathring{H}^{1,2}(B). \quad (8.1)$$

Proof: We consider the continuous embeddings

$$\mathring{H}^{1,2}(B) \hookrightarrow L^q(B) \hookrightarrow L^2(B),$$

for any $q \geq 2$, where the first one is compact due to Sobolev's embedding theorem. Hence, we may apply Ehrling's interpolation lemma, yielding

$$\|\varphi\|_{L^q(B)} \leq \epsilon \,\|\varphi\|_{\mathring{H}^{1,2}(B)} + C(q, \epsilon) \qquad \forall\, \varphi \in S\mathring{H}^{1,2}(B),$$

for any $\epsilon > 0$ and any $q \geq 2$, where we used the requirement $\|\varphi\|_{L^2(B)} = 1$. Hence, together with Hölder's, Cauchy-Schwarz' and Poincaré's inequalities we achieve for any $\epsilon > 0$:

$$\begin{aligned}
\|\, KE\,\varphi^2 \,\|_{L^1(B)} &\leq \|\, KE \,\|_{L^{p^*}(B)} \|\varphi\|^2_{L^{2p'}(B)} \\
&\leq \|\, KE \,\|_{L^{p^*}(B)} \,(\epsilon \,\|\varphi\|_{\mathring{H}^{1,2}(B)} + C(p', \epsilon))^2 \\
\leq \|\, KE \,\|_{L^{p^*}(B)} \, 2\,(\epsilon^2 (C_P + 1) &\int_B |\nabla\varphi|^2 \; dw + C(p', \epsilon)^2),
\end{aligned}$$

with $\frac{1}{p^*}+\frac{1}{p'}=1$, and therefore by the definition of J:

$$J(\varphi) \geq (1-4 \parallel KE \parallel_{L^{p^*}(B)} (C_P+1)\,\epsilon^2) \int_B |\nabla\varphi|^2 \,dw$$
$$-4 \parallel KE \parallel_{L^{p^*}(B)} C(p',\epsilon)^2,$$

for any $\varphi \in S\mathring{H}^{1,2}(B)$, which yields our assertion by a suitable choice of ϵ.

$\diamond$

Now we can prove

Theorem 8.1 *The spectra of A and $\bar{A}$ coincide, are discrete and accumulate only at ∞, thus their eigenspaces are finite dimensional. Furthermore there holds for their common smallest eigenvalue $\lambda_{\min}$:*

$$\lambda_{\min}(A) = \inf_{S\mathrm{Dom}(A)} J = \inf_{S\mathring{H}^{1,2}(B)} J = \inf_{S(H^{2,2}(B)\cap\mathring{H}^{1,2}(B))} J = \lambda_{\min}(\bar{A}). \quad (8.2)$$

Proof: Firstly the above proposition guarantees the existence of $\inf_{S\mathring{H}^{1,2}(B)} J$. Hence, we may consider some sequence $\{\varphi_j\} \subset S\mathring{H}^{1,2}(B)$ such that $J(\varphi_j) \searrow \inf_{S\mathring{H}^{1,2}(B)} J$, and again using (8.1) we conclude together with Poincaré's inequality that $\parallel \varphi_j \parallel_{H^{1,2}(B)} \leq$ *const.*. Thus we can extract some subsequence $\{\varphi_{j_k}\}$ such that

$$\varphi_{j_k} \rightharpoonup \varphi^* \qquad \text{weakly in} \quad H^{1,2}(B),$$

for some $\varphi^* \in \mathring{H}^{1,2}(B)$. Since this implies $\varphi_{j_k} \longrightarrow \varphi^*$ in $L^q(B)$, for any $q \geq 1$, we infer $\varphi^* \in S\mathring{H}^{1,2}(B)$. Furthermore this implies:

$$\parallel KE\,(\varphi_{j_k}^2 - (\varphi^*)^2) \parallel_{L^1(B)} \leq \parallel KE \parallel_{L^{p^*}(B)} \parallel \varphi_{j_k}^2 - (\varphi^*)^2 \parallel_{L^{p'}(B)} \longrightarrow 0, \quad (8.3)$$

with $\frac{1}{p^*}+\frac{1}{p'}=1$. Hence, J inherits the weak lower semicontinuity of the Dirichlet integral:

$$J(\varphi^*) = \int_B |\nabla\varphi^*|^2 + 2\,(KE)^\tau (\varphi^*)^2\,dw$$
$$\leq \liminf_{k\to\infty} \int_B |\nabla\varphi_{j_k}|^2 \,dw + 2 \lim_{k\to\infty} \int_B KE\,\varphi_{j_k}^2\,dw$$
$$= \liminf_{k\to\infty} J(\varphi_{j_k}) = \inf_{S\mathring{H}^{1,2}(B)} J, \quad (8.4)$$

thus $J(\varphi^*) = \inf_{S\mathring{H}^{1,2}(B)} J$. Now we construct recursively a filtration of subspaces $\mathring{H}^{1,2}(B) =: U_1 \supset U_2 \supset U_3 \ldots$ of $\mathring{H}^{1,2}(B)$ by

$$U_i := \{\eta \in \mathring{H}^{1,2}(B) \mid \langle \eta, \varphi_j^* \rangle_{L^2(B)} = 0, \;\; j = 1,\ldots,i-1\}, \quad (8.5)$$

for $i \geq 2$, and $SU_i := U_i \cap S\mathring{H}^{1,2}(B)$, where we set $\varphi_1^* := \varphi^*$ and the $\varphi_i^* \in SU_i$ have to minimize J:

$$J(\varphi_i^*) \stackrel{!}{=} \inf_{SU_i} J =: \lambda_i. \tag{8.6}$$

We obtain those minimizers φ_i^*, $i \geq 2$, exactly by the same procedure which yielded φ^* above since the U_i's are closed w. r. to weak $H^{1,2}(B)$-convergence and non-trivial, otherwise there would hold $\mathrm{Span}(\varphi_1^*, \ldots, \varphi_{i-1}^*)^\perp = \{0\}$ ($\perp$ w. r. to $\langle \cdot\,, \cdot \rangle_{L^2(B)}$ in $\mathring{H}^{1,2}(B)$) which contradicts $\dim \mathring{H}^{1,2}(B) = \infty$ due to the projection theorem. By construction of our filtration the sequence $\{\lambda_i\}$ is increasing. Furthermore $\{\infty\}$ is its only point of accumulation since if there were a bounded subsequence $\{\lambda_{i_k}\}$ then we would conclude by (8.6), (8.1) and Poincaré's inequality that $\| \varphi_{i_k}^* \|_{H^{1,2}(B)} \leq$const. $\forall\, k \in \mathbb{N}$. Hence, since the embedding $H^{1,2}(B) \hookrightarrow L^2(B)$ is compact, $\{\varphi_{i_k}^*\}$ would possess a Cauchy-subsequence w. r. to $\| \cdot \|_{L^2(B)}$, which contradicts the fact that

$$\langle \varphi_i^* - \varphi_j^*, \varphi_i^* - \varphi_j^* \rangle_{L^2(B)} = \| \varphi_i^* \|^2_{L^2(B)} - 2\, \langle \varphi_i^*, \varphi_j^* \rangle_{L^2(B)} + \| \varphi_j^* \|^2_{L^2(B)} = 2 - 2\delta_{ij},$$

$\forall\, i, j \in \mathbb{N}$, due to $\varphi_i^* \in SU_i$ and the construction in (8.5). Now we are going to prove that the φ_i^* and λ_i are indeed eigenfunctions and eigenvalues of A and $\bar{A}$. For some fixed i we consider an arbitrary $\psi \in U_i$ and the function

$$f_i(\epsilon) := J(\varphi_i^* + \epsilon\psi) - \lambda_i \, \| \varphi_i^* + \epsilon\psi \|^2_{L^2(B)} \qquad \text{on} \quad [-\epsilon_0, \epsilon_0],$$

for $\epsilon_0 > 0$ that small such that $\| \varphi_i^* + \epsilon\psi \|_{L^2(B)} > 0 \;\; \forall\, \epsilon \in [-\epsilon_0, \epsilon_0]$. Since J is a quadratic form and

$$J\Big(\frac{\varphi_i^* + \epsilon\psi}{\| \varphi_i^* + \epsilon\psi \|_{L^2(B)}} \Big) \geq \inf_{SU_i} J = \lambda_i$$

we have that $f_i \geq 0$ on $[-\epsilon_0, \epsilon_0]$ and $f_i(0) = J(\varphi_i^*) - \lambda_i = 0$ by (8.6). Thus we obtain for any $\psi \in U_i$ and any $i \in \mathbb{N}$, abbreviating $\langle \cdot\,, \cdot \rangle := \langle \cdot\,, \cdot \rangle_{L^2(B)}$:

$$\begin{aligned} 0 = \frac{d}{d\epsilon} f_i(\epsilon)\, |_{\epsilon=0} = 2 \int_B \nabla\varphi_i^* \cdot \nabla\psi + 2\,(KE)^\tau \varphi_i^* \,\psi \, dw - 2\,\lambda_i \,\langle \varphi_i^*, \psi \rangle \\ = 2\,(\mathcal{L}(\varphi_i^*, \psi) - \lambda_i \,\langle \varphi_i^*, \psi \rangle). \end{aligned} \tag{8.7}$$

Now we have to verify this equation for some arbitrary $\psi \in \mathring{H}^{1,2}(B)$, which is already the case for $i = 1$. To this end we consider the "coordinates" $c_j := \langle \varphi_j^*, \psi \rangle_{L^2(B)}$ of ψ w. r. to the φ_j^*'s, set $\psi_{i-1} := \sum_{j=1}^{i-1} c_j \, \varphi_j^*$ for $i \geq 2$ and see that $\psi - \psi_{i-1} \in U_i$ by:

$$\langle \varphi_k^*, \psi - \psi_{i-1} \rangle = \langle \varphi_k^*, \psi \rangle - \langle \varphi_k^*, \sum_{j=1}^{i-1} c_j \, \varphi_j^* \rangle = c_k - \sum_{j=1}^{i-1} c_j \delta_{kj} = 0,$$

for $k = 1, \ldots, i-1$. Now combining this with $\langle \varphi_i^*, \varphi_j^* \rangle = 0$ for $j = 1, \ldots, i-1$ by (8.5), the symmetry of $\mathcal{L}(\cdot\,, \cdot)$, $\varphi_i^* \in U_i \subset U_j$ for $j = 1, \ldots, i-1$ and

(8.7) we obtain indeed:

$$\begin{aligned}\mathcal{L}(\varphi_i^*,\psi)-\lambda_i\langle\varphi_i^*,\psi\rangle &= \Big(\mathcal{L}(\varphi_i^*,\psi)-\mathcal{L}(\varphi_i^*,\psi_{i-1})+\sum_{j=1}^{i-1}c_j\,\mathcal{L}(\varphi_i^*,\varphi_j^*)\Big)\\ &\quad-\Big(\lambda_i\langle\varphi_i^*,\psi\rangle-\lambda_i\langle\varphi_i^*,\psi_{i-1}\rangle+\sum_{j=1}^{i-1}c_j\,\lambda_j\langle\varphi_j^*,\varphi_i^*\rangle\Big)\\ &=(\mathcal{L}(\varphi_i^*,\psi-\psi_{i-1})-\lambda_i\langle\varphi_i^*,\psi-\psi_{i-1}\rangle)\\ &\quad+\sum_{j=1}^{i-1}c_j\,(\mathcal{L}(\varphi_j^*,\varphi_i^*)-\lambda_j\,\langle\varphi_j^*,\varphi_i^*\rangle)=0,\end{aligned}$$

for any $\psi\in\mathring{H}^{1,2}(B)$ and $i\geq 2$, i.e.

$$A(\varphi_i^*)=\lambda_i\,\varphi_i^*\qquad\text{weakly on } B, \tag{8.8}$$

$\forall\, i\in\mathbb{N}$. Now we know that our coefficients $2(KE)^\tau-\lambda_i$ are of class $C^\infty(B)$ for any $\tau\in K(\tilde f)$ (see below (2.38)). Thus the L^2-regularity theory, Theorem 8.13 in [8], yields that $\varphi_i^*\in C^\infty(B)$, $\forall\, i\in\mathbb{N}$. Hence, if we test (8.8) with an arbitrary $\psi\in C_c^\infty(B)$ and apply the divergence theorem to $\nabla\varphi_i^*\,\psi\in C_c^\infty(B)$, then we obtain:

$$\langle A(\varphi_i^*),\psi\rangle=\mathcal{L}(\varphi_i^*,\psi)=\lambda_i\,\langle\varphi_i^*,\psi\rangle.$$

Thus the fundamental lemma of the calculus of variations yields the equation (8.8) even in the classical sense on B. In particular we see that $\varphi_i^*\in Dom(A)$, thus indeed the φ_i^*'s and the λ_i's are eigenfunctions and eigenvalues of A and therefore also of $\bar{A}$, $\forall\, i\in\mathbb{N}$. Now we are going to prove

$$\|\,\psi\,\|^2_{L^2(B)}=\sum_{j=1}^{\infty}c_j^2 \tag{8.9}$$

for any $\psi\in\mathring{H}^{1,2}(B)$ and $c_j=\langle\varphi_j^*,\psi\rangle$. We gain by $\psi-\psi_i\in U_{i+1}$ and (8.5):

$$\langle\psi_i,\psi-\psi_i\rangle=\sum_{j=1}^{i}c_j\,\langle\varphi_j^*,\psi-\psi_i\rangle=0$$

and together with (8.8) also:

$$\mathcal{L}(\psi_i,\psi-\psi_i)=\sum_{j=1}^{i}c_j\,\lambda_j\langle\varphi_j^*,\psi-\psi_i\rangle=0,$$

$\forall\, i\in\mathbb{N}$, which implies

$$\begin{aligned}&\|\,\psi-\psi_i\,\|^2_{L^2(B)}=\langle\psi,\psi-\psi_i\rangle=\|\,\psi\,\|^2_{L^2(B)}-\|\,\psi_i\,\|^2_{L^2(B)}\\ &\text{and}\qquad J(\psi-\psi_i)=\mathcal{L}(\psi,\psi-\psi_i)=J(\psi)-J(\psi_i).\end{aligned} \tag{8.10}$$

Due to $\lambda_i \nearrow \infty$ we can now choose some $N \in \mathbb{N}$ that large such that $\lambda_N > 0$ and obtain by (8.8):

$$J(\psi_i) = J\Big(\sum_{j=1}^{i} c_j\, \varphi_j^*\Big) = \sum_{j,k=1}^{i} c_j\, c_k\, \mathcal{L}(\varphi_j^*, \varphi_k^*) = \sum_{j,k=1}^{i} c_j\, c_k\, \lambda_j\, \delta_{jk}$$
$$= \sum_{j=1}^{i} \lambda_j\, c_j^2 \geq \sum_{j=1}^{N} \lambda_j\, c_j^2 = J(\psi_N),$$

for any $i \geq N$. Hence, also noting $J(\psi - \psi_i) \geq \lambda_{i+1} \parallel \psi - \psi_i \parallel^2_{L^2(B)}$ due to (8.6) and $\psi - \psi_i \in U_{i+1}$ we achieve together with (8.10):

$$0 \leq \parallel \psi \parallel^2_{L^2(B)} - \parallel \psi_i \parallel^2_{L^2(B)} = \parallel \psi - \psi_i \parallel^2_{L^2(B)} \leq \frac{1}{\lambda_{i+1}}\, J(\psi - \psi_i)$$
$$= \frac{1}{\lambda_{i+1}}\, (J(\psi) - J(\psi_i)) \leq \frac{1}{\lambda_{i+1}}\, (J(\psi) - J(\psi_N)) \longrightarrow 0,$$

for $i \to \infty$, thus indeed by $\psi_i = \sum_{j=1}^{i} c_j\, \varphi_j^*$:

$$\parallel \psi \parallel^2_{L^2(B)} = \lim_{i \to \infty} \parallel \psi_i \parallel^2_{L^2(B)} = \lim_{i \to \infty} \sum_{j,k=1}^{i} c_j\, c_k\, \delta_{jk} \equiv \sum_{j=1}^{\infty} c_j^2.$$

Now we suppose that $\lambda \notin \{\lambda_i\}$ is a further eigenvalue of $\bar{A}$ and $\phi \in ES_\lambda(\bar{A})$ a corresponding eigenfunction. Since $\phi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B) = Dom(\bar{A})$ by Theorem 7.2 we have $\nabla\phi\, \psi \in \mathring{H}^{1,1}(B)$ for any $\psi \in C_c^\infty(B)$. Hence, we may apply the divergence theorem for Sobolev functions to $\nabla\phi\, \psi$ in order to obtain

$$\mathcal{L}(\phi, \psi) = \langle \bar{A}(\phi), \psi \rangle = \lambda\, \langle \phi, \psi \rangle, \tag{8.11}$$

and we achieve this equality also for any $\psi \in \mathring{H}^{1,2}(B)$ exactly as in the proof of Prop. 7.4 by approximation. Now testing this weak equation with $\psi := \varphi_i^*$ for an arbitrary $i \in \mathbb{N}$ we conclude together with (8.8):

$$\lambda\, \langle \phi, \varphi_i^* \rangle = \mathcal{L}(\phi, \varphi_i^*) = \mathcal{L}(\varphi_i^*, \phi) = \lambda_i\, \langle \varphi_i^*, \phi \rangle,$$

hence, $0 = (\lambda - \lambda_i)\, \langle \varphi_i^*, \phi \rangle = (\lambda - \lambda_i)\, c_i$, $\forall\, i \in \mathbb{N}$, which would imply that all the coordinates c_i of ϕ would vanish. Thus together with (8.9) we would obtain:

$$0 = \sum_{j=1}^{\infty} c_j^2 = \parallel \phi \parallel^2_{L^2(B)},$$

in contradiction to the assumption that $\phi \not\equiv 0$ is an eigenfunction. Hence, indeed we proved $\{\lambda_i\} = Spec(\bar{A})$, which yields $\{\lambda_i\} \subset Spec(A) \subset Spec(\bar{A}) = \{\lambda_i\}$ and therefore also $\{\lambda_i\} = Spec(A)$. Finally we note that we have due to $Dom(A) \subset Dom(\bar{A}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\varphi^* \in SDom(A)$ and (8.4):

$$\inf_{S\mathring{H}^{1,2}(B)} J \leq \inf_{S(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))} J \leq \inf_{SDom(A)} J \leq J(\varphi^*) = \inf_{S\mathring{H}^{1,2}(B)} J,$$

which together with $\inf_{S\mathring{H}^{1,2}(B)} J = \lambda_1 = \lambda_{\min}(A) = \lambda_{\min}(\bar{A})$ completes also the proof of (8.2).

$\diamond$

We abbreviate $\lambda_{\min} := \lambda_{\min}(A) = \lambda_{\min}(\bar{A})$ and prove

Theorem 8.2 *(i) For an eigenfunction $\varphi^* \in \mathrm{ES}_{\lambda_{\min}}(\bar{A})$ there holds $\mid \varphi^* \mid > 0$ on B and therefore:*

$$\dim \mathrm{ES}_{\lambda_{\min}}(\bar{A}) = \dim \mathrm{ES}_{\lambda_{\min}}(A) = 1. \tag{8.12}$$

(ii) Especially an eigenfunction $\varphi^* \in \mathrm{ES}_{\lambda_{\min}}(A)$ satisfies $\mid \varphi^* \mid > 0$ on B.

Proof: Let $\varphi^* \in ES_{\lambda_{\min}}(\bar{A}) \subset H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ with $\| \varphi^* \|_{L^2(B)} = 1$ be given arbitrarily. Firstly we note that $\mid \varphi^* \mid \in \mathring{H}^{1,2}(B)$ and that

$$\int_B \mid \nabla \mid \varphi^* \mid\mid^2 \, dw = \int_B \mid \nabla \varphi^* \mid^2 \, dw. \tag{8.13}$$

Moreover as (8.11) holds true for any $\phi \in Dom(\bar{A})$ and any $\psi \in \mathring{H}^{1,2}(B)$ we may apply (8.11) to $\phi := \varphi^* =: \psi$. Thus together with (8.13) and (8.2) we achieve:

$$\begin{aligned} J(\mid \varphi^* \mid) = J(\varphi^*) = \langle \bar{A}(\varphi^*), \varphi^* \rangle_{L^2(B)} &= \lambda_{\min} \langle \varphi^*, \varphi^* \rangle_{L^2(B)} \\ &= \lambda_{\min} = \inf_{S\mathring{H}^{1,2}(B)} J. \end{aligned}$$

Hence, exactly as we obtained (8.8) we achieve now due to $\mid \varphi^* \mid \in \mathring{H}^{1,2}(B)$:

$$A(\mid \varphi^* \mid) = \lambda_{\min} \mid \varphi^* \mid \qquad \text{weakly on } B.$$

Thus on account of $KE \in C^\infty(B)$ (see below (2.38)) we achieve $\mid \varphi^* \mid \in C^\infty(B)$ by Theorem 8.13 in [8] and moreover we may apply Harnack's inequality, Theorem 8.20 in [8], to $\mid \varphi^* \mid \geq 0$ on any disc $B_{4R}(\bar{w}) \subset\subset B$:

$$\sup_{B_R(\bar{w})} \mid \varphi^* \mid \leq const. \inf_{B_R(\bar{w})} \mid \varphi^* \mid . \tag{8.14}$$

Hence, if we had $\varphi^*(w_0) = 0$ for some arbitrary point $w_0 \in B_R(\bar{w})$ we could conclude now that $\varphi^* \equiv 0$ on $B_R(\bar{w})$, and a standard argument making a successive use of Harnack's inequality (8.14) would show that $\varphi^* \equiv 0$ on B, which contradicts our assumption $\| \varphi^* \|_{L^2(B)} = 1$. Thus we have proved indeed for an arbitrary eigenfunction $\varphi^* \in ES_{\lambda_{\min}}(\bar{A})$ that $\varphi^*(w) > 0$ or $< 0 \;\forall\, w \in B$. Now we assume that $\dim ES_{\lambda_{\min}}(\bar{A}) > 1$. On account of the projection theorem we could choose two $\mathrm{L}^2(B)$-orthogonal eigenfunctions $\varphi^*, \bar{\varphi}^*$ in $ES_{\lambda_{\min}}(\bar{A})$, i.e. with $\langle \varphi^*, \bar{\varphi}^* \rangle_{L^2(B)} = 0$, in contradiction to $\langle \varphi^*, \bar{\varphi}^* \rangle_{L^2(B)} > 0$ or < 0. As we have $\{0\} \neq ES_{\lambda_{\min}}(A) \subset ES_{\lambda_{\min}}(\bar{A})$ points (i) and (ii) of the theorem are proved at once.

$\diamond$

Chapter 9

The component $K(\tilde{f})^1_{\tau^*}$ of $K(f)$ is a closed $C^\omega-$ curve

We shall prove the main result, Theorem 1.1, by contradiction. Thus we assume the existence of some $X^* \in \mathcal{M}_s(\Gamma)$ and some sequence $\{X^n\} \subset \mathcal{M}(\Gamma)$ with

$$X^n \longrightarrow X^* \qquad in\ C^0(\bar{B}, \mathbb{R}^3). \tag{9.1}$$

Thus by means of (2.5) the points $\tau^* := \psi^{-1}(X^*) \in K_s(f)$ and $\tau^n := \psi^{-1}(X^n) \in K(f)$ satisfy

$$\tau^n \longrightarrow \tau^* \qquad \text{in } K(f), \tag{9.2}$$

where we introduced the notation

$$K_s(f) := \psi^{-1}(\mathcal{M}_s(\Gamma)).$$

By $K(f) \subset K(\tilde{f})$ τ^* would be a non-isolated critical point of $\tilde{f}$ and therefore $\text{rank}(D^2\tilde{f}(\tau^*)) \leq N-1$. Moreover we know that $X(\,\cdot\,, \tau^*) \equiv \tilde{\psi}(\tau^*)$ coincides with $\psi(\tau^*) = X^* \in \mathcal{M}_s(\Gamma)$ by Corollary 2.4. Hence, we have $\kappa(\tau^*) = 0$ and could conclude now by Heinz' formula (2.15) $\dim \text{Ker } A^{X(\,\cdot\,,\tau^*)} \geq 1$, thus 0 would be an eigenvalue of $A^{\tau^*} := A^{X(\,\cdot\,,\tau^*)}$. Moreover we know together with Lemma 3.1 that there holds

$$J^{\tau^*} := J^{X(\,\cdot\,,\tau^*)} \geq 0 \qquad \text{on } \mathring{H}^{1,2}(B), \tag{9.3}$$

thus in particular on $\text{Dom}(A^{\tau^*})$. Therefore 0 would even be the smallest eigenvalue of A^{τ^*}. For if there were some negative eigenvalue $\lambda^* < 0$ of A^{τ^*} with some eigenfunction φ^*, we would obtain by the proof of the symmetry of A^{τ^*} in Proposition 7.4 that

$$J^{\tau^*}(\varphi^*) = \mathcal{L}^{\tau^*}(\varphi^*, \varphi^*) = \langle A^{\tau^*}(\varphi^*), \varphi^* \rangle_{L^2(B)} = \lambda^* \langle \varphi^*, \varphi^* \rangle_{L^2(B)} < 0, \tag{9.4}$$

which is a contradiction. Hence, together with (8.12) we would arrive at

$$\dim Ker(A^{\tau^*}) \equiv \dim ES_{\lambda_{\min}=0}(A^{\tau^*}) = 1. \tag{9.5}$$

In combination with $\kappa(\tau^*) = 0$ we could therefore derive from formula (2.15) exactly $\text{rank}(D^2\tilde{f}(\tau^*)) = N-1$. Hence, there would be a $\delta > 0$ such that

$$\text{rank}(D^2\tilde{f}) \geq N-1 \quad \text{on} \quad B_\delta(\tau^*), \tag{9.6}$$

where we abbreviate $B_\delta(\tau^*)$ for $B_\delta^N(\tau^*) \cap \mathbb{R}^N \subset\subset T$. Due to $\tilde{f} \in C^\omega(T)$ we know that $K(\tilde{f})$ is an analytic set which possesses therefore a locally finite analytic triangulation due to [27], p. 463, and is therefore especially locally connected, such that a combination of (9.6) and (9.2) would lead to the

Contradiction hypothesis :
If the assertion of the main result, Theorem 1.1, were wrong, then there would have to exist some point $\tau^* \in K_s(f)$ which is also contained in the 1-skeleton $K(\tilde{f})^1$ of the analytic simplicial complex $K(\tilde{f})$, i.e. $\tau^* \in K_s(f) \cap K(\tilde{f})^1 \neq \emptyset$.

Now this gives rise to the idea to analyze the following subset Z of the connected component $K(\tilde{f})^1_{\tau^*}$ of the 1-skeleton $K(\tilde{f})^1$ (of $K(\tilde{f})$) that contains τ^*:

$$Z := \{\tau \in K(\tilde{f})^1_{\tau^*} \mid \kappa(\tau) = 0,\ \text{rank}(D^2(\tilde{f})(\tau)) = N-1, J^\tau \geq 0 \text{ on } C_c^\infty(B)\}.$$

Now we prove the following crucial

Theorem 9.1 *The set Z $(\neq \emptyset)$ is an open and closed subset of $K(\tilde{f})^1_{\tau^*}$, thus $Z = K(\tilde{f})^1_{\tau^*}$.*

9.1 Proof of Theorem 9.1

(a) By $\tau^* \in Z$ we know that Z is not empty.
b) Secondly we derive the "openness" of the condition $\kappa(\tau) = 0$ by Corollary 4.4:

Theorem 9.2 *Let $\bar{\tau} \in Z$ be some arbitrarily fixed point, then there exists some $\delta > 0$ such that $\kappa \equiv 0$ on $B_\delta(\bar{\tau}) \cap K(\tilde{f})$.*

Proof: Since we have rank $D^2(\tilde{f})(\bar{\tau}) = N-1$ there exists some $\delta > 0$ such that rank $D^2(\tilde{f})(\tau) \geq N-1$ for any $\tau \in B_\delta(\bar{\tau})$. Hence, together with Heinz' formula (2.15) we can conclude that $2\,\kappa(\tau) \leq 1$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})$. Thus recalling Corollary 4.4 we achieve in fact $\kappa(\tau) = 0$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})$.

◇

c) Next we show that $\kappa(\tau) = 0$ and $J^\tau \geq 0$ on $C_c^\infty(B)$ are "closed conditions". To this end we combine Theorem 9.2 with Corollary 4.3 in order to achieve

Corollary 9.1 *Let $\bar{\tau} \in Z$ be some arbitrarily fixed point, then there exists some $\delta > 0$ such that there holds $K(\tilde{f}) \cap B_\delta(\bar{\tau}) = K(f) \cap B_\delta(\bar{\tau})$. In particular, this shows $Z \subset K(f)$.*

Proof: The inclusion "$\supset$" follows from Corollary 2.4. "$\subset$": By Theorem 9.2 we know that there exists some $\delta > 0$ such that there holds $\kappa \equiv 0$ on $K(\tilde{f}) \cap B_\delta(\bar{\tau})$. Now if the assertion were wrong there would have to exist some point $\tau^* \in K(\tilde{f}) \cap B_\delta(\bar{\tau})$ which is contained in $K(\tilde{f}) \setminus K(f)$ and therefore $X(\,\cdot\,,\tau^*) \in \tilde{\mathcal{M}}(\Gamma) \setminus \mathcal{M}(\Gamma)$ on account of Lemma 2.1 and points (v) and (iii) of Theorem 2.1. By Corollary 4.3 this implies that $X(\,\cdot\,,\tau^*)$ would have to possess a boundary branch point in contradiction to $\kappa(\tau^*) = 0$. Now the second assertion follows from the first one due to $Z \subset K(\tilde{f})$ by its definition.

$\diamond$

Now we consider a sequence $\{\tau^n\} \subset Z$ that converges to some point $\hat{\tau} \in K(\tilde{f})^1_{\tau^*}$. On account of Corollary 9.1 and the closedness of $K(f)$ we have $\tau^n \to \hat{\tau}$ in $K(f)$. Thus by the properties of the points of Z, Corollary 2.4, (2.5) and Theorems 3.1 and 4.1, i.e. by the closedness of $\mathcal{M}_s(\Gamma)$, we see:

$$X(\,\cdot\,,\tau^n) = \psi(\tau^n) \longrightarrow \psi(\hat{\tau}) = X(\,\cdot\,,\hat{\tau}) \qquad \textit{in } (\mathcal{M}_s(\Gamma), \|\cdot\|_{C^0(\bar{B})}), \tag{9.7}$$

which means that $\kappa(\hat{\tau}) = 0$ and $J^{\hat{\tau}} \geq 0$ on $C_c^\infty(B)$, proving the closedness of the conditions $\kappa(\tau) = 0$ and $J^\tau \geq 0$ on $C_c^\infty(B)$.

d) Now we show the openness of the condition $\operatorname{rank}(D^2\tilde{f}) = N - 1$. As already used in (9.6) we achieve for any fixed point $\bar{\tau} \in Z$ the existence of some $\delta > 0$ such that

$$\operatorname{rank}(D^2\tilde{f}) \geq N - 1 \qquad \text{on } B_\delta(\bar{\tau}). \tag{9.8}$$

Now since $K(\tilde{f})^1_{\tau^*}$ is a piecewise analytic curve none of its points can be an isolated critical point of $\tilde{f}$, which implies in fact by (9.8):

$$\operatorname{rank}(D^2\tilde{f}) \equiv N - 1 \qquad \text{on } B_\delta(\bar{\tau}) \cap K(\tilde{f})^1_{\tau^*}. \tag{9.9}$$

Moreover the condition $\operatorname{rank}(D^2\tilde{f}) = N-1$ is also closed, for let $\{\tau^n\} \subset Z$ be some sequence converging to some point $\hat{\tau} \in K(\tilde{f})^1_{\tau^*}$, then $\operatorname{rank}(D^2\tilde{f}(\hat{\tau})) \leq N - 1$, since otherwise $\hat{\tau}$ would be an isolated critical point of $\tilde{f}$. Inserting this and $\kappa(\hat{\tau}) = 0$ (by (c)) into formula (2.15) we see that 0 is an eigenvalue of $A^{\hat{\tau}}$. Thus as we also know $J^{\hat{\tau}} \geq 0$ on $\operatorname{Dom}(A^{\hat{\tau}})$ by (c) and Lemma 3.1 we gain as in (9.4) that 0 is even the smallest eigenvalue of $A^{\hat{\tau}}$ and therefore as in (9.5):

$$\dim ES_{\lambda_{\min}=0}(A^{\hat{\tau}}) = \dim Ker(A^{\hat{\tau}}) = 1.$$

Hence, inserting this and $\kappa(\hat{\tau}) = 0$ into Heinz' formula (2.15) again we achieve exactly $\operatorname{rank}(D^2(\tilde{f})(\hat{\tau})) = N - 1$.
e) Finally we prove the openness of the stability condition, i.e. of $J^\tau \geq 0$ on $C_c^\infty(B)$. To this end let

$$\iota : (H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \|\cdot\|_{H^{2,2}(B)}) \hookrightarrow (L^2(B), \|\cdot\|_{L^2(B)})$$

denote the inclusion, thus $\text{image}(\iota) = Dom(\bar{A}^\tau)$ for any $\tau \in K(\tilde{f})$,

$$\mathcal{S}(H^{2,2}(B) \cap \mathring{H}^{1,2}(B)) := \{\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B) \mid \| \varphi \|_{H^{2,2}(B)} = 1\}$$

and $\|\cdot\|$ the operator norm for operators L mapping $(H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \|\cdot\|_{H^{2,2}(B)})$ into $L^2(B)$, i.e.

$$\| L \| := \sup\{\| L(\varphi) \|_{L^2(B)} \mid \varphi \in \mathcal{S}(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))\}.$$

Now we will combine Corollary 4.2, Theorem 9.2 and the proofs of Prop. 7.1 and Corollary 7.1 in order to achieve

Proposition 9.1 *For any fixed $\bar{\tau} \in Z$ there exists some $\bar{\delta} > 0$ and some constant $C(\alpha)$ only depending on α, $\bar{\tau}$ and $\bar{\delta}$ such that there holds*

$$| (\mathrm{KE})^\tau \varphi(w) | \leq C(\alpha) \sum_{k=1}^{N+3} | w - e^{i\tau_k} |^{-1+\frac{\alpha}{2}} \| \dot{\triangle} \varphi \|_{L^2(B)} \qquad \forall\, w \in B, \tag{9.10}$$

for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ and any $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$.

Proof: Theorem 9.2 guarantees the existence of some neighborhood $B_\delta(\bar{\tau})$ such that $X(\cdot, \tau)$ is free of branch points on $\bar{B}$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(f)$. Thus we obtain the existence of some $\bar{\delta} \in (0, \delta]$ such that estimate (4.18) holds for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ with some constant C that does not depend on τ. Applying this in the ending of the proof of Prop. 7.1 we achieve estimate (9.10) for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ and any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and then even for any $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ by the proof of Corollary 7.1.

$\diamond$

Now using this central estimate we are going to prove the crucial

Theorem 9.3 *Let $\bar{\tau}$ be some arbitrary point of Z. Then there holds for an arbitrary sequence $\{\tau^n\} \subset K(\tilde{f})$ with $\tau^n \longrightarrow \bar{\tau}$:*

$$\| \bar{A}^{\tau^n} \circ \iota - \bar{A}^{\bar{\tau}} \circ \iota \| \longrightarrow 0 \qquad \text{for } n \longrightarrow \infty.$$

Proof: Suppose the assertion would be wrong. Set

$$S_n := \sup\{\| ((KE)^{\tau^n} - (KE)^{\bar\tau})\varphi \|_{L^2(B)} |\ \varphi \in \mathcal{S}(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))\}$$
$$= \| \bar A^{\tau^n} \circ \iota - \bar A^{\bar\tau} \circ \iota \|,$$

and let $\{\epsilon_n\}$ be an arbitrary null-sequence. By the definition of the supremum there exists for each $n \in \mathbb{N}$ some function $\varphi_n \in \mathcal{S}(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))$ that satisfies

$$0 \le S_n - \| ((KE)^{\tau^n} - (KE)^{\bar\tau})\varphi_n \|_{L^2(B)} < \epsilon_n. \tag{9.11}$$

We set $g_n := ((KE)^{\tau^n} - (KE)^{\bar\tau})\varphi_n$ for each n. Firstly we infer from the requirement that $\{\tau^n\} \subset K(\tilde f)$ converges to $\bar\tau \in Z$ that $\kappa(\tau^n) = \kappa(\bar\tau) = 0$ and that τ^n and $\bar\tau$ are contained in $K(f)$ for sufficiently large $n > N$ on account of Theorem 9.2 and Corollary 9.1. Thus we can see by Sobolev's embedding theorem due to $2 - \frac{2}{2} = 1$ and $\| \varphi_n \|_{H^{2,2}(B)} = 1$ in combination with Corollary 2.4, (2.5), Cauchy's estimates and (2.37):

$$| g_n(w) | \le \| \varphi_n \|_{L^\infty(B)} | (KE)^{\tau^n} - (KE)^{\bar\tau} | (w)$$
$$\le const. \mid (KE)^{\tau^n} - (KE)^{\bar\tau} \mid (w) \longrightarrow 0 \qquad \textit{for } n \to \infty, \tag{9.12}$$

pointwise for any $w \in B$. Furthermore on account of the facts that τ^n and $\bar\tau$ are contained in $K(f)$ for $n > N$ and $\tau^n \to \bar\tau$ we can apply Prop. 9.1 and obtain together with $\| \triangle\varphi_n \|_{L^2(B)} \le 1$ the estimate

$$| g_n(w) | \le | (KE)^{\tau^n}\varphi_n | (w) + | (KE)^{\bar\tau}\varphi_n | (w)$$
$$\le C(\alpha)\Big(\sum_{k=1}^{N+3} | w - e^{i\tau^n_k} |^{-1+\frac{\alpha}{2}} + | w - e^{i\bar\tau_k} |^{-1+\frac{\alpha}{2}} \Big), \tag{9.13}$$

for any $w \in B$ and $n > \bar N$, with $\bar N$ sufficiently large. Now we verify the requirements of Vitali's theorem applied to $\{g_n\}$. To this end let $E \subset B$ be an arbitrary measurable subset with positive $\mathcal{L}^2$-measure and define $R := \sqrt{\mathcal{L}^2(E)}$. Now $\tau^n \to \bar\tau$ implies in particular the existence of some number $d > 0$ such that $\operatorname{dist}(\tau^n, \partial T) > d$, $\forall\, n \in \mathbb{N}$. Thus we can conclude that there has to exist some $\bar R > 0$ such that

$$2\,\bar R < \min_{k=1,\dots,N+3} \{| e^{i\tau^n_{k+1}} - e^{i\tau^n_k} |, | e^{i\bar\tau_{k+1}} - e^{i\bar\tau_k} |\},$$

with $\tau_{N+4} := \tau_1$, uniformly $\forall\, n \in \mathbb{N}$. Then we obtain the following estimate:

$$\|| w - e^{i\tau^n_k} |^{-1+\frac{\alpha}{2}}\|^2_{L^2(E)} \le \int_{\bigcup_{j=1}^{N+3} B_R(e^{i\tau^n_j})} | w - e^{i\tau^n_k} |^{-2+\alpha}\ dw$$
$$+ \int_{E \setminus \bigcup_{j=1}^{N+3} B_R(e^{i\tau^n_j})} | w - e^{i\tau^n_k} |^{-2+\alpha}\ dw$$
$$\le (N+2)R^{-2+\alpha}\pi R^2 + 2\pi \int_0^R r^{-2+\alpha}\, r\, dr + \mathcal{L}^2(E)R^{-2+\alpha}$$
$$= \big((N+2)\,\pi + \frac{2\,\pi}{\alpha} + 1\big)\, R^\alpha \longrightarrow 0$$

for $\bar{R} > R \searrow 0$, i.e. for $\mathcal{L}^2(E) \searrow 0$, uniformly for any $n \in \mathbb{N}$ and for any $k = 1, \ldots, N+3$. Thus by the same estimate for the summands in (9.13) involving the $\bar{\tau}_k$ and by Minkowski's inequality we achieve finally

$$\| g_n \|_{L^2(E)} \longrightarrow 0 \qquad \textit{if} \quad \mathcal{L}^2(E) \searrow 0,$$

uniformly for any $n > \bar{N}$. Hence, together with (9.12) Vitali's theorem yields $\| g_n \|_{L^2(B)} \longrightarrow 0$, for $n \to \infty$, and therefore together with (9.11):

$$0 \le S_n < \| g_n \|_{L^2(B)} + \epsilon_n \longrightarrow 0 \qquad \textit{for} \quad n \to \infty.$$

$\diamond$

Furthermore using that

$$\bar{A}^\tau \circ \iota : (H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \| \cdot \|_{H^{2,2}(B)}) \longrightarrow (L^2(B), \| \cdot \|_{L^2(B)})$$

are bounded operators due to Prop. 7.3 and by image$(\iota) = Dom(\bar{A}^\tau)$, for any $\tau \in K(\tilde{f})$, we can immediately conclude from the above theorem and Theorem 2.29 on p. 207 in Kato's book [24] that

$$\bar{A}^{\tau^n} \longrightarrow \bar{A}^{\bar{\tau}} \qquad \text{in the generalized sense,} \tag{9.14}$$

if $K(\tilde{f}) \ni \tau^n \longrightarrow \bar{\tau}$, for an arbitrarily fixed point $\bar{\tau} \in Z$, where we used Kato's terminology in [24], p. 202. Now since $\bar{A}^\tau$ has a discrete spectrum only accumulating at ∞, for any $\tau \in K(\tilde{f})$, by Theorem 8.1 we can apply Theorem 3.16 on pp. 212–213 in [24] which yields due to (9.14):

Corollary 9.2 *Let $\bar{\tau}$ be some arbitrary point of Z, $\{\tau^n\} \subset K(\tilde{f})$ an arbitrary sequence with $\tau^n \longrightarrow \bar{\tau}$ and $d \in \mathbb{R} \setminus \operatorname{Spec}(\bar{A}^{\bar{\tau}})$ arbitrarily fixed. Then there holds also $d \in \mathbb{R} \setminus \operatorname{Spec}(\bar{A}^{\tau^n})$ and*

$$\dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\tau^n}) \equiv \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\bar{\tau}})$$

for sufficiently large n.

Now together with (b), (d) and formulas (2.15), (8.2) and (8.12) we can prove

Theorem 9.4 *For any $\bar{\tau} \in Z$ there is some neighborhood $B_\epsilon(\bar{\tau})$ such that $J^\tau \ge 0$ on $Dom(A^\tau)$ for any $\tau \in B_\epsilon(\bar{\tau}) \cap K(\tilde{f})^1_{\tau^*}$.*

Proof: We suppose the assertion to be wrong, i.e. that there exists some sequence $\{\tau^n\} \subset K(\tilde{f})^1_{\tau^*}$ with $\tau^n \longrightarrow \bar{\tau}$ and $\inf_{Dom(A^{\tau^n})} J^{\tau^n} < 0 \quad \forall\, n \in \mathbb{N}$. Hence, by (8.2) we achieve:

$$\lambda_{\min}(\bar{A}^{\tau^n}) = \lambda_{\min}(A^{\tau^n}) = \inf_{SDom(A^{\tau^n})} J^{\tau^n} < 0 \qquad \forall\, n \in \mathbb{N}. \tag{9.15}$$

Now by (b) and (d) we know already that $\bar{\tau}$ possesses some neighborhood $B_\delta(\bar{\tau})$ such that there hold $\kappa(\tau) = 0$ and $\operatorname{rank}(D^2(\tilde{f})(\tau)) = N - 1$ $\forall\, \tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})^1_{\tau^*}$. Hence, in combination with Heinz' formula (2.15) we conclude that $\operatorname{Ker}(A^{\tau^n}) \neq \{0\}$, thus by $Dom(A^{\tau^n}) \subset Dom(\bar{A}^{\tau^n})$ that $\operatorname{Ker}(\bar{A}^{\tau^n}) \neq \{0\}$ for $n > \bar{n}$ and some sufficiently large $\bar{n}$. Therefore we achieve together with (9.15):

$$\dim \bigoplus_{\lambda \leq 0} ES_\lambda(\bar{A}^{\tau^n}) \geq 2 \qquad \forall\, n > \bar{n}. \tag{9.16}$$

Now we know for $\bar{\tau} \in Z$ by the definition of Z and formula (2.15) that $\dim Ker(A^{\bar{\tau}}) = 1$, thus especially that 0 is an eigenvalue of $A^{\bar{\tau}}$, and since Lemma 3.1 yields $J^{\bar{\tau}} \geq 0$ on $\operatorname{Dom}(A^{\bar{\tau}})$ we can conclude as in (9.4) that 0 is even the smallest eigenvalue of $A^{\bar{\tau}}$. Hence, we infer from formula (8.12) that

$$\dim ES_{\lambda_{\min}=0}(\bar{A}^{\bar{\tau}}) = \dim ES_{\lambda_{\min}=0}(A^{\bar{\tau}}) = 1. \tag{9.17}$$

Now on account of $K(\tilde{f}) \ni \tau^n \longrightarrow \bar{\tau}$ we can apply Corollary 9.2 with $d := \frac{\lambda_1(\bar{A}^{\bar{\tau}}) - \lambda_{\min}(\bar{A}^{\bar{\tau}})}{2} > 0$, which yields together with (9.16), $\lambda_{\min}(\bar{A}^{\bar{\tau}}) = 0$ and (9.17):

$$2 \leq \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\tau^n}) \equiv \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\bar{\tau}}) = \dim ES_0(\bar{A}^{\bar{\tau}}) = 1$$

for sufficiently large n, which is a contradiction.

$\diamond$

Hence, in fact $Z \neq \emptyset$ turns out to be an open and closed subset of the connected set $K(\tilde{f})^1_{\tau^*}$, thus $Z = K(\tilde{f})^1_{\tau^*}$.

$\diamondsuit$

Next combining the above corollary with the result $Z = K(\tilde{f})^1_{\tau^*}$ and the implicit function theorem for real analytic functions, [5] p. 268, we finally achieve

Corollary 9.3 *The set $Z = K(\tilde{f})^1_{\tau^*}$ is a <u>closed</u> analytic curve.*

Proof: Firstly we know that $K(\tilde{f})$ is a closed subset of T and therefore also its 1-skeleton $K(\tilde{f})^1$ and its connected component $K(\tilde{f})^1_{\tau^*}$. Moreover we know that $Z = K(\tilde{f})^1_{\tau^*}$ is contained in $K(f) \subset\subset T$, by Corollary 9.1, which yields the closedness of the set Z w. r. to the standard topology of $\mathbb{R}^N$ and therefore its compactness. Thus together with the fact that the analytic set $K(\tilde{f})$ possesses a locally finite analytic triangulation $Z = K(\tilde{f})^1_{\tau^*}$ can only consist of a finite number of consecutive analytic arcs. Now we show that the set Z is not only a piecewise but an entirely analytic curve, i.e. it does not have any "corners". Firstly there exists some $\delta > 0$ such that $K(\tilde{f})^1_{\tau^*} \cap B_\delta(\bar{\tau})$ is connected on account of the local connectedness of $K(\tilde{f})^1_{\tau^*}$. Moreover we fix some point $\bar{\tau} \in Z$ and derive from its property $\mathrm{rank}(D^2\tilde{f}(\bar{\tau})) = N-1$ and from the symmetry of $D^2\tilde{f}(\bar{\tau})$ the existence of a uniquely determined permutation of the coordinates $\tau_1, \dots, \tau_N$ in T such that there holds

$$\det D_{\hat{\tau}}(\nabla_{\hat{\tau}}(\tilde{f}))(\hat{\bar{\tau}}, \bar{\tau}_N) \neq 0,$$

where we denote by $\hat{\tau} := (\tau_1, \dots, \tau_{N-1})$ the tuple of the first $N-1$ permuted coordinates. Hence we can choose the above δ that small, depending on $\bar{\tau}$, such that there holds

$$\det D_{\hat{\tau}}(\nabla_{\hat{\tau}}(\tilde{f}))(\hat{\tau}, \tau_N) \neq 0 \qquad \forall\, (\hat{\tau}, \tau_N) = \tau \in B_\delta(\bar{\tau}). \tag{9.18}$$

Hence, we obtain by the implicit function theorem for analytic functions, [5] p. 268, applied to $\nabla_{\hat{\tau}}\tilde{f} \in C^\omega(T, \mathbb{R}^{N-1})$ that

$$M_\delta(\bar{\tau}) := \{(\hat{\tau}, \tau_N) = \tau \in B_\delta(\bar{\tau}) \mid \nabla_{\hat{\tau}}(\tilde{f})(\hat{\tau}, \tau_N) = 0\} \tag{9.19}$$

is a one dimensional analytic submanifold of $B_\delta(\bar{\tau})$, containing $K(\tilde{f})^1_{\tau^*} \cap B_\delta(\bar{\tau})$ in particular. Thus we can conclude that $K(\tilde{f})^1_{\tau^*} \cap B_\delta(\bar{\tau})$ is a one dimensional connected analytic manifold, possibly with boundary, which proves that $Z = K(\tilde{f})^1_{\tau^*}$ is a one dimensional compact connected analytic manifold, possibly with boundary, as the point $\bar{\tau}$ was chosen arbitrarily in Z. Hence, we infer from the classification theorem of one dimensional compact connected smooth manifolds (see the appendix in [10]) that Z is either homeomorphic to $[0,1]$ or $\mathbb{S}^1$. Now we suppose $Z \cong [0,1]$ and consider some boundary point $\bar{\tau} \in \partial Z$. We note that there holds $\nabla_{\hat{\tau}}(\tilde{f})(\bar{\tau}) = 0$ and (9.18). Now fixing some sufficiently small $\tilde{\delta} \in (0, \delta]$ the implicit function theorem for analytic functions yields the existence of some neighborhood $J := [\bar{\tau}_N - \epsilon_1, \bar{\tau}_N + \epsilon_2]$ of $\bar{\tau}_N$, depending on $\tilde{\delta}$, and some C^ω-map $g : J \longrightarrow \mathbb{R}^{N-1}$ such that $g(\bar{\tau}_N) = \hat{\bar{\tau}}$ and

$$\mathit{graph}(g \mid_{\mathring{J}}) := \{(g(\tau_N), \tau_N) \mid \tau_N \in \mathring{J}\} = M_{\tilde{\delta}}(\bar{\tau}). \tag{9.20}$$

Hence, recalling definition (9.19) we conclude immediately that

$$\mathit{graph}(g \mid_{\mathring{J}}) \supset Z \cap B_{\tilde{\delta}}(\bar{\tau}). \tag{9.21}$$

Now the continuity and injectivity of $(g(\,\cdot\,), \,\cdot\,)$ on J implies that $(g(\,\cdot\,), \,\cdot\,) : \mathring{J} \xrightarrow{\cong} \mathit{graph}(g \mid_{\mathring{J}})$ performs a homeomorphism. Hence, since $Z \cap B_{\tilde{\delta}}(\bar{\tau})$ is connected we conclude that $(g(\,\cdot\,), \,\cdot\,)^{-1}(Z \cap B_{\tilde{\delta}}(\bar{\tau}))$ is connected as well and therefore an intervall I. Moreover we infer from $\bar{\tau} \in \partial Z$ and $(g(\bar{\tau}_N), \bar{\tau}_N) = \bar{\tau}$ that $\bar{\tau}_N \in \partial I$. Thus we have either $I \subset (\bar{\tau}_N - \epsilon_1, \bar{\tau}_N]$ or $I \subset [\bar{\tau}_N, \bar{\tau}_N + \epsilon_2)$ and we shall assume the first case without loss of generality. Then we infer especially that

$$\frac{\partial \tilde{f}}{\partial \tau_N}(g(\tau_N), \tau_N) = 0 \qquad \forall\, \tau_N \in I \tag{9.22}$$

by $Z = K(\tilde{f})^1_{\tau^*}$. Moreover since $\frac{\partial \tilde{f}}{\partial \tau_N}(g(\,\cdot\,),\,\cdot\,)$ is analytic on J we can conclude by the identity theorem for real analytic functions that (9.22) extends in fact onto J, i.e.

$$\frac{\partial \tilde{f}}{\partial \tau_N}(g(\tau_N), \tau_N) = 0 \qquad \forall\, \tau_N \in J.$$

Now together with (9.19) and (9.20) this implies firstly $(g(\tau_N), \tau_N) \in K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau})\ \ \forall\, \tau_N \in \mathring{J}$. Next we know that $K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau})$ is contained in the one dimensional manifold $M_{\tilde{\delta}}(\bar{\tau})$ implying $K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau}) = K(\tilde{f})^1 \cap B_{\tilde{\delta}}(\bar{\tau})$, and thus we obtain

$$\mathit{graph}(g\mid_{\mathring{J}}) \subset K(\tilde{f})^1_{\tau^*} \cap B_{\tilde{\delta}}(\bar{\tau}) = Z \cap B_{\tilde{\delta}}(\bar{\tau}),$$

where we used that $(g(\bar{\tau}_N), \bar{\tau}_N) = \bar{\tau}$ is contained in the connected component $K(\tilde{f})^1_{\tau^*}$ of τ^* and thus the entire graph of $g\mid_{\mathring{J}}$. Together with (9.21) and the definition of I we obtain therefore:

$$\mathit{graph}(g\mid_{\mathring{J}}) = Z \cap B_{\tilde{\delta}}(\bar{\tau}) = \mathit{graph}(g\mid_I).$$

Hence, we can infer that for any point $\tau_N^2 \in (\bar{\tau}_N, \bar{\tau}_N + \epsilon_2)$ there would have to exist some point $\tau_N^1 \in I \subset (\bar{\tau}_N - \epsilon_1, \bar{\tau}_N]$ such that $(g(\tau_N^1), \tau_N^1) = (g(\tau_N^2), \tau_N^2)$, thus especially $\tau_N^1 = \tau_N^2$, which contradicts $\tau_N^1 \leq \bar{\tau}_N < \tau_N^2$ and proves in fact $Z \cong \mathbb{S}^1$.

$\diamond$

Chapter 10

Strict monotonicity of Tomi's function $\mathcal{F}(X^{(\,\cdot\,)})$

Now the implicit function theorem (in its C^ω-version) yields an analytic regular parametrization $\tilde{\tau}:[0,2\pi]/(0\sim 2\pi)\longrightarrow Z$ of the analytic closed curve $Z=K(\tilde{f})^1_{\tau^*}$, which corresponds via ψ to the closed path $X^t := X(\,\cdot\,,\tilde{\tau}(t))$ of minimal surfaces in $\mathcal{M}_s(\Gamma)$, where we recall that ψ and $\tilde{\psi}$ coincide on $K(f)\supset Z$ by Corollary 2.4. Following an idea due to Tomi in [35] and [36] we are going to consider the composition of the so-called volume functional $\mathcal{F}$ (up to a factor $\frac{1}{3}$) with this path of minimal surfaces X^t, i.e.

$$\mathcal{F}(X^t) := \int_B \langle X^t_u \wedge X^t_v, X^t\rangle\, dw, \tag{10.1}$$

which we shall term Tomi's function and whose existence is guaranteed by estimate (10.2) below. Just as Tomi did we aim to derive its strict monotonicity on $[0,2\pi]$, which contradicts $\tilde{\tau}(0)=\tilde{\tau}(2\pi)$ as a result of our *contradiction hypothesis* in Chapter 10 and thus proves Theorem 1.1.
We fix some $\bar{\tau}\in T$ and $l\in\{1,\ldots,N\}$ arbitrarily and infer from Theorem 2.1 (vii), (2.19) and (2.20) that there exists some $\delta>0$ such that there hold the estimates

$$\mid X_w(w,\tau)\mid \leq \mathit{const.}(\delta,\bar{\tau},k)\ \mid w-e^{i\tau_k}\mid^{\rho}, \tag{10.2}$$

for any $k\in\{1,\ldots,N+3\}$ and for $k\neq l$:

$$\mid X_{\tau_l}(w,\tau)\mid \leq \mathit{const.}(\delta,\bar{\tau},l,k)\mid w-e^{i\tau_k}\mid^{\rho+1}, \tag{10.3}$$

$\forall\, w\in B_\delta(e^{i\bar{\tau}_k})\cap B$ and $\forall\,\tau\in B_\delta(\bar{\tau}):=B^N_\delta(\bar{\tau})\cap\mathbb{R}^N\subset\subset T$, but for $k=l$ only

$$\mid X_{\tau_l}(w,\tau)\mid \leq \mathit{const.}(\delta,\bar{\tau},l)\mid w-e^{i\tau_l}\mid^{\rho}, \tag{10.4}$$

which we shall avoid in the sequel by using (2.22), thus the estimate

$$\mid X_{\tau_l}(w,\tau)+X_\varphi(w,\tau)\mid \leq \mathit{const.}(\delta,\bar{\tau},l)\mid w-e^{i\tau_l}\mid^{\rho+1}, \tag{10.5}$$

$\forall\, w \in B_\delta(e^{i\bar\tau_l}) \cap B$ and $\forall\, \tau \in B_\delta(\bar\tau)$, where we abbreviate $\rho := \min_{k=1,\dots,N+3} \rho_1^k \in (-1,0]$ for the smallest exponent of the ρ_j^k, for $j = 1,\dots,p_k$ and $k = 1,\dots,N+3$. Now we consider the functions

$$\begin{aligned} q^l(w,\tau) &:= \langle X_u(w,\tau) \wedge X_v(w,\tau), X_{\tau_l}(w,\tau)\rangle \\ &= \langle X_u(w,\tau) \wedge X_v(w,\tau), X_{\tau_l}(w,\tau) + X_\varphi(w,\tau)\rangle \end{aligned}$$

for $l = 1,\dots,N$, $w \in \bar B \setminus \{e^{i\tau_k}\}$ and $\tau \in T$, where we used that

$$\langle X_u \wedge X_v, X_\varphi\rangle(w) = u\,\langle X_u \wedge X_v, X_v\rangle(w) - v\,\langle X_u \wedge X_v, X_u\rangle(w) = 0$$

for any $X \in C^1(B,\mathbb{R}^3)$ and $w \in B$. Moreover we infer from (10.2)–(10.5):

$$\mid q^l(w,\tau) \mid\leq c(\delta,\bar\tau,l,k) \mid w - e^{i\tau_k} \mid^{2\rho}, \tag{10.6}$$

$\forall\, w \in B_\delta(e^{i\bar\tau_k}) \cap B$, $k = 1,\dots,N+3$, and $\forall\, \tau \in B_\delta(\bar\tau)$. In the sequel we will denote $\triangle_\sigma(\bar\tau) := B \setminus \bigcup_{k=1}^{N+3} \overline{B_\sigma(e^{i\bar\tau_k})}$, for $\sigma < \delta$, and prove for any fixed $l = 1,\dots,N$:

Proposition 10.1 *The function $Q^l(\,\cdot\,) := \int_B q^l(w,\,\cdot\,)\,dw$ is continuous on T.*

Proof: Firstly we can infer from estimate (10.6) that the function Q^l exists in any point $\bar\tau$ of T. Now we fix some $\sigma \in (0,\delta)$ and $\bar\tau \in T$ arbitrarily and introduce the functions $Q^l_\sigma(\tau) := \int_{\triangle_\sigma(\bar\tau)} q^l(w,\tau)\,dw$, for $\tau \in B_\delta(\bar\tau) \subset\subset T$, which are well-defined again due to estimate (10.6). We have:

$$\mid Q^l_\sigma(\bar\tau) - Q^l_\sigma(\tau) \mid\leq \int_{\triangle_\sigma(\bar\tau)} \mid q^l(w,\bar\tau) - q^l(w,\tau) \mid\, dw. \tag{10.7}$$

By Hilfssatz 1 (A) in [17] $X_{\tau_l}(\,\cdot\,,\,\cdot\,)$ is uniformly continuous on $\overline{\triangle_\sigma(\bar\tau)} \times \overline{B_{\frac{\sigma}{2}}(\bar\tau)}$ due to $e^{i\tau_k} \in \overline{B_{\frac{\sigma}{2}}(e^{i\bar\tau_k})}$ for $\mid \bar\tau - \tau \mid\leq \frac{\sigma}{2}$, $\forall\, k \in \{1,\dots,N+3\}$. Hence, for every $\epsilon > 0$ there is some $\varrho(\epsilon) > 0$ such that

$$\mid X_{\tau_l}(w,\bar\tau) - X_{\tau_l}(w,\tau) \mid< \epsilon,$$

if $\mid (w,\bar\tau) - (w,\tau) \mid< \varrho$, i.e. if $\mid \bar\tau - \tau \mid< \varrho$ uniformly for any $w \in \overline{\triangle_\sigma(\bar\tau)}$, which means that

$$X_{\tau_l}(\,\cdot\,,\tau) \longrightarrow X_{\tau_l}(\,\cdot\,,\bar\tau) \qquad in\ C^0(\overline{\triangle_\sigma(\bar\tau)}), \tag{10.8}$$

for $\tau \to \bar\tau$. Furthermore we know by Hilfssatz 1 (A) in [17] resp. point (vi) of Theorem 2.1 that $D_{(u,v)}X(\,\cdot\,,\,\cdot\,)$ is uniformly continuous on $\overline{\triangle_\sigma(\bar\tau)} \times \overline{B_{\frac{\sigma}{2}}(\bar\tau)}$ again due to $e^{i\tau_k} \in \overline{B_{\frac{\sigma}{2}}(e^{i\bar\tau_k})}$ for $\mid \bar\tau - \tau \mid\leq \frac{\sigma}{2}$, $\forall\, k \in \{1,\dots,N+3\}$. Hence, as above we obtain that

$$D_{(u,v)}X(\,\cdot\,,\tau) \longrightarrow D_{(u,v)}X(\,\cdot\,,\bar\tau) \qquad in\ C^0(\overline{\triangle_\sigma(\bar\tau)}), \tag{10.9}$$

for $\tau \to \bar{\tau}$. Thus combining (10.8) and (10.9) we infer

$$q^l(\,\cdot\,,\tau) \longrightarrow q^l(\,\cdot\,,\bar{\tau}) \qquad in\ C^0(\overline{\triangle_\sigma(\bar{\tau})}),$$

which yields together with Lebesgue's convergence theorem and (10.7):

$$\mid Q^l_\sigma(\bar{\tau}) - Q^l_\sigma(\tau) \mid \leq \int_{\triangle_\sigma(\bar{\tau})} \mid q^l(w,\bar{\tau}) - q^l(w,\tau) \mid \, dw \longrightarrow 0, \tag{10.10}$$

for $\tau \to \bar{\tau}$ and any fixed $\sigma < \delta$. Moreover we obtain by estimate (10.6) for any $\sigma \in (0,\delta)$ and $\tau \in B_\delta(\bar{\tau})$:

$$\begin{aligned} \mid Q^l(\tau) - Q^l_\sigma(\tau) \mid =\mid \int_B q^l(w,\tau)\, dw - \int_{\triangle_\sigma(\bar{\tau})} q^l(w,\tau)\, dw \mid \\ \leq \sum_{k=1}^{N+3} \int_{B_\sigma(e^{i\bar{\tau}_k})\cap B} \mid q^l(w,\tau) \mid \, dw \\ \leq \sum_{k=1}^{N+3} c(\delta,\bar{\tau},l,k) \int_{B_\sigma(e^{i\bar{\tau}_k})\cap B} \mid w - e^{i\tau_k} \mid^{2\rho} \, dw. \end{aligned} \tag{10.11}$$

Now we estimate for any $k = 1,\ldots,N+3$ and $\sigma \in (0,\delta)$:

$$\begin{aligned} \int_{B_\sigma(e^{i\bar{\tau}_k})} \mid w - e^{i\tau_k} \mid^{2\rho} \, dw \\ \leq \int_{B_\sigma(e^{i\tau_k})} \mid w - e^{i\tau_k} \mid^{2\rho} \, dw + \int_{B_\sigma(e^{i\bar{\tau}_k})\setminus B_\sigma(e^{i\tau_k})} \mid w - e^{i\tau_k} \mid^{2\rho} \, dw \\ \leq 2\pi \int_0^\sigma r^{2\rho+1}\, dr + \mid B_\sigma(e^{i\bar{\tau}_k}) \setminus B_\sigma(e^{i\tau_k}) \mid \sigma^{2\rho} \leq \frac{\rho+2}{\rho+1}\,\pi\,\sigma^{2\rho+2}. \end{aligned}$$

Hence, together with (10.11) and $2\rho + 2 > 0$ we achieve:

$$\mid Q^l(\tau) - Q^l_\sigma(\tau) \mid \leq (N+3)\ const.(\delta,\bar{\tau},l)\, \frac{\rho+2}{\rho+1}\,\pi\,\sigma^{2\rho+2} \longrightarrow 0, \tag{10.12}$$

for $\sigma \searrow 0$, uniformly in $\tau \in B_\delta(\bar{\tau})$. Now we split:

$$\begin{aligned} \mid Q^l(\bar{\tau}) - Q^l(\tau) \mid \\ \leq \mid Q^l(\bar{\tau}) - Q^l_\sigma(\bar{\tau}) \mid + \mid Q^l_\sigma(\bar{\tau}) - Q^l_\sigma(\tau) \mid + \mid Q^l_\sigma(\tau) - Q^l(\tau) \mid \end{aligned} \tag{10.13}$$

for any $\sigma \in (0,\delta)$ and $\tau \in B_\delta(\bar{\tau})$. We choose some $\epsilon > 0$ arbitrarily and obtain by (10.12) the existence of some $\bar{\sigma}(\epsilon) \in (0,\delta)$ such that

$$\mid Q^l(\tau) - Q^l_{\bar{\sigma}}(\tau) \mid < \frac{\epsilon}{3} \qquad \text{uniformly } \forall\, \tau \in B_\delta(\bar{\tau}). \tag{10.14}$$

Next we know by (10.10) applied to $\sigma := \bar{\sigma}(\epsilon)$ that there exists some $\bar{\delta}(\epsilon) \in (0,\delta)$ such that

$$\mid Q^l_{\bar{\sigma}}(\bar{\tau}) - Q^l_{\bar{\sigma}}(\tau) \mid < \frac{\epsilon}{3} \qquad \forall\, \tau \in B_{\bar{\delta}}(\bar{\tau}). \tag{10.15}$$

Hence, combining (10.13), (10.14) and (10.15) we achieve that for any $\epsilon > 0$ there exists some sufficiently small $\bar{\delta}(\epsilon) \in (0, \delta)$ such that

$$\begin{aligned} &\mid Q^l(\bar{\tau}) - Q^l(\tau) \mid \\ \leq &\mid Q^l(\bar{\tau}) - Q^l_{\bar{\sigma}}(\bar{\tau}) \mid + \mid Q^l_{\bar{\sigma}}(\bar{\tau}) - Q^l_{\bar{\sigma}}(\tau) \mid + \mid Q^l_{\bar{\sigma}}(\tau) - Q^l(\tau) \mid < 3\,\frac{\epsilon}{3} = \epsilon, \end{aligned}$$

if $\tau \in B_{\bar{\delta}}(\bar{\tau})$, which proves the continuity of Q^l in $\bar{\tau}$, for the arbitrarily chosen point $\bar{\tau} \in T$, and thus its continuity on T.

$\diamond$

Due to the analyticity of $\tilde{\tau}$ and $\frac{\partial}{\partial t} X^t = \sum_{l=1}^N \frac{d\tilde{\tau}_l}{dt} X^t_{\tau_l}$ the above proposition implies in particular that the integral

$$\Phi_1(t) := \int_{B_1(0)} \Big\langle X^t_u \wedge X^t_v, \frac{\partial}{\partial t} X^t \Big\rangle \, dw \tag{10.16}$$

depends continuously on $t \in [0, 2\pi]/(0 \sim 2\pi)$. Now we are going to prove

Proposition 10.2 *There holds*

$$\int_{\partial B_r(0)} \Big\langle X^t_\varphi \wedge X^t, \frac{\partial}{\partial t} X^t \Big\rangle \, ds \longrightarrow 0 \qquad \text{in } C^0([0, 2\pi]), \tag{10.17}$$

for $r \nearrow 1$.

Proof: We consider for some fixed $l \in \{1, \ldots, N\}$ the functions

$$\begin{aligned} h^l_r(\varphi, \tau) &:= \langle X_\varphi(re^{i\varphi}, \tau) \wedge X(re^{i\varphi}, \tau), X_{\tau_l}(re^{i\varphi}, \tau) \rangle \\ &\equiv \langle X_\varphi(re^{i\varphi}, \tau) \wedge X(re^{i\varphi}, \tau), X_{\tau_l}(re^{i\varphi}, \tau) + X_\varphi(re^{i\varphi}, \tau) \rangle, \end{aligned} \tag{10.18}$$

for $r \in (0, 1)$, $\tau \in T$ and $\varphi \in [0, 2\pi]/(0 \sim 2\pi)$. Again we fix some $\bar{\tau} \in T$ arbitrarily. Firstly we derive a uniform bound for $\mid X(\,\cdot\,,\,\cdot\,) \mid$ on $B \times \overline{B_\varepsilon(\bar{\tau})}$ for some $\varepsilon > 0$. We know that there exists some $\delta > 0$ such that there hold (2.17) and (10.2) on $B_\delta(e^{i\bar{\tau}_k}) \times B_\delta(\bar{\tau})$ for $k = 1, \ldots, N+3$. Moreover we may apply Theorem 2.1 (vi) to the domain $D := \triangle_{\frac{\delta}{2}}(\bar{\tau})$ which yields a uniform bound $b(\delta, \bar{\tau})$ of $\mid X_w(\,\cdot\,,\,\cdot\,) \mid$ on $\overline{\triangle_\delta(\bar{\tau})} \times \overline{B_\varepsilon(\bar{\tau})}$ for any $\varepsilon \in (0, \frac{\delta}{2})$ due to $e^{i\tau_k} \in B_{\frac{\delta}{2}}(e^{i\bar{\tau}_k})$ for $\mid \bar{\tau} - \tau \mid \leq \varepsilon < \frac{\delta}{2}$. Hence, we can estimate for some arbitrarily chosen k:

$$\begin{aligned} \mid X(0, \tau) - P_k \mid = \Big| \int_0^1 \frac{d}{dt} X(te^{i\tau_k}, \tau) \, dt \Big| &\leq \int_0^1 \mid DX(te^{i\tau_k}, \tau) \mid \, dt \\ &= \int_0^{1-\delta} 2 \mid X_w(te^{i\tau_k}, \tau) \mid \, dt + \int_{1-\delta}^1 2 \mid X_w(te^{i\tau_k}, \tau) \mid \, dt \\ &\leq 2\, b(\delta, \bar{\tau})\, (1 - \delta) + \int_{1-\delta}^1 2\, c(\delta, \bar{\tau}, k) \mid te^{i\tau_k} - e^{i\tau_k} \mid^\rho \, dt \\ &= 2\, b(\delta, \bar{\tau})\, (1 - \delta) + 2\, c(\delta, \bar{\tau}, k) \int_0^\delta y^\rho \, dy \\ &= 2 \Big(b(\delta, \bar{\tau})\, (1 - \delta) + c(\delta, \bar{\tau}, k) \frac{\delta^{\rho+1}}{\rho + 1} \Big), \end{aligned}$$

and therefore

$$
\begin{aligned}
| X(w,\tau) | &\leq | X(0,\tau) | + \left| \int_0^1 \frac{d}{dt} X(t\,w,\tau)\,dt \right| \\
&\leq | X(0,\tau) | + \int_0^1 | DX(t\,w,\tau) || w |\; dt \\
&\leq | P_k | + 2\Big(b(\delta,\bar\tau)\,(1-\delta) + c(\delta,\bar\tau,k)\frac{\delta^{\rho+1}}{\rho+1} + b(\delta,\bar\tau)\Big),
\end{aligned}
$$

for $(w,\tau) \in \overline{\triangle_\delta(\bar\tau)} \times \overline{B_\varepsilon(\bar\tau)}$ and an arbitrarily chosen k. Hence, together with (2.17) on $B_\delta(e^{i\bar\tau_k}) \times B_\delta(\bar\tau)$, for $k = 1,\dots,N+3$, and $\rho + 1 > 0$ we achieve the desired uniform bound for $| X(\,\cdot\,,\,\cdot\,) |$ on $B \times \overline{B_\varepsilon(\bar\tau)}$ for any $\varepsilon \in (0, \frac{\delta}{2})$. Moreover we obtain by Hilfssatz 1 (A) in [17] that $X_{\tau_l}(\,\cdot\,,\,\cdot\,)$ is uniformly continuous on $\overline{\triangle_\delta(\bar\tau)} \times \overline{B_\varepsilon(\bar\tau)}$ for any $\varepsilon \in (0,\frac{\delta}{2})$, in particular uniformly bounded, due to $e^{i\tau_k} \in B_{\frac{\delta}{2}}(e^{i\bar\tau_k})$ for $| \bar\tau - \tau | \leq \varepsilon < \frac{\delta}{2}$, $\forall\, k \in \{1,\dots,N+3\}$. Hence, together with (10.3) we have proved the existence of some $\delta > 0$ such that $| X_{\tau_l}(\,\cdot\,,\,\cdot\,) |$ is uniformly bounded on $(B \setminus B_\delta(e^{i\bar\tau_l})) \times \overline{B_\varepsilon(\bar\tau)}$ for any $\varepsilon \in (0,\frac{\delta}{2})$. Thus taking also (10.2) and (10.5) into account we conclude that there holds for any $\varepsilon \in (0,\frac{\delta}{2})$:

$$
| h^l_r(\varphi,\tau) | \leq \mathit{const.}(\delta,\bar\tau,l) \sum_{k=1}^{N+3} | re^{i\varphi} - e^{i\tau_k} |^\rho, \tag{10.19}
$$

$\forall\, r \in (0,1)$, $\forall\, \varphi \in [0,2\pi]$ and $\forall\, \tau \in B_\varepsilon(\bar\tau)$, where we abbreviate $\rho := \min_{k=1,\dots,N+3}\{\rho^k_1\} \in (-1,0]$ for the smallest exponent of the ρ^k_j for $j = 1,\dots,p_k$ and $k = 1,\dots,N+3$. Now we estimate $| re^{i\varphi} - e^{i\tau_k} |^\rho$ independently of $r \in (0,1)$. To this end we fix some $\tau \in B_\varepsilon(\bar\tau)$, $k \in \{1,\dots,N+3\}$, $r \in (0,1)$ and $\varphi \in [0,2\pi] \setminus \{\tau_k\}$ and choose some $R \in (0,\frac{1}{8})$ arbitrarily. Now there are two possibilities:
(I) There holds $| re^{i\varphi} - e^{i\tau_k} | < R$ or (II) $| re^{i\varphi} - e^{i\tau_k} | \geq R$.
Case (I): We consider the angle $\gamma := | \mathit{angle}(e^{i\tau_k} - e^{i\varphi}, re^{i\varphi} - e^{i\varphi}) |$ which depends on the fixed φ only and note that $\gamma \in (\frac{\pi}{4},\frac{\pi}{2})$ by the requirement of Case (I) and $\varphi \neq \tau_k$. Now we compute

$$
\begin{aligned}
&| re^{i\varphi} - e^{i\tau_k} |^2 \qquad (10.20)\\
&= | re^{i\varphi} - e^{i\varphi} |^2 + | e^{i\varphi} - e^{i\tau_k} |^2 - 2\,| re^{i\varphi} - e^{i\varphi} || e^{i\varphi} - e^{i\tau_k} |\,\cos(\gamma)\\
&= (1-r)^2 + | e^{i\varphi} - e^{i\tau_k} |^2 - 2\,(1-r)\,| e^{i\varphi} - e^{i\tau_k} |\,\cos(\gamma)
\end{aligned}
$$

and consider this expression as a quadratic function of $1-r$, i.e. look at

$$
q(x) := x^2 + y^2 - 2\,x\,y\,\cos(\gamma) = (x - y\cos(\gamma))^2 + (y\,\sin(\gamma))^2
$$

for $x \in [0,1]$ and $y := | e^{i\varphi} - e^{i\tau_k} |$. Obviously there holds $q \geq (y\,\sin(\gamma))^2$ on $[0,1]$, thus by (10.20)

$$
| re^{i\varphi} - e^{i\tau_k} | = \sqrt{q(1-r)} \geq | e^{i\varphi} - e^{i\tau_k} |\,\sin(\gamma),
$$

and therefore by $\rho \in (-1,0]$:

$$| re^{i\varphi} - e^{i\tau_k} |^\rho \leq | e^{i\varphi} - e^{i\tau_k} |^\rho \ \sin(\gamma)^\rho. \tag{10.21}$$

Moreover we recall that $| e^{i\varphi} - e^{i\tau_k} | = 2\sin\Big(\frac{|\varphi-\tau_k|}{2}\Big)$. Now by $\sin(\theta) \geq \frac{\theta}{2}$ for $\theta \in [0,\frac{\pi}{2}]$ and $\frac{\pi}{2} \geq \frac{|\varphi-\tau_k|}{2}$, we have

$$\sin\Big(\frac{| \varphi - \tau_k |}{2}\Big) \geq \frac{| \varphi - \tau_k |}{4}. \tag{10.22}$$

Furthermore we gain by $\gamma \in (\frac{\pi}{4},\frac{\pi}{2})$ that $\sin(\gamma) > \frac{1}{\sqrt{2}} > \frac{1}{2}$ and therefore together with (10.21) and (10.22):

$$| re^{i\varphi} - e^{i\tau_k} |^\rho \leq \Big(2\sin\Big(\frac{| \varphi - \tau_k |}{2}\Big)\Big)^\rho \sin(\gamma)^\rho < \Big(\frac{| \varphi - \tau_k |}{4}\Big)^\rho,$$

for every $\varphi \in [0,2\pi] \setminus \{\tau_k\}$ and $k \in \{1,\dots,N+3\}$. On the other hand in Case (II) we have $| re^{i\varphi} - e^{i\tau_k} |^\rho \leq R^\rho$. Hence, we achieve in any case the estimate

$$| re^{i\varphi} - e^{i\tau_k} |^\rho \leq \Big(\frac{| \varphi - \tau_k |}{4}\Big)^\rho + R^\rho, \tag{10.23}$$

for any $k \in \{1,\dots,N+3\}$, and therefore together with (10.19):

$$| h_r^l(\varphi,\tau) | \leq c(\delta,\bar\tau,l) \sum_{k=1}^{N+3} \Big(\Big(\frac{| \varphi - \tau_k |}{4}\Big)^\rho + R^\rho\Big), \tag{10.24}$$

$\forall\, r \in (0,1)$, $\forall\, \varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\dots,N+3}$, $\forall\, \tau \in B_\varepsilon(\bar\tau)$, yielding a Lebesgue dominating term for the family $\{h_r^l(\,\cdot\,,\tau)\}_{r\in(0,1)}$ in $L^1([0,2\pi])$ for every $\tau \in B_\varepsilon(\bar\tau)$ on account of $\rho > -1$. Furthermore we conclude from the property $X(e^{i\varphi},\tau) \in \Gamma_j$ for $\varphi \in [\tau_j,\tau_{j+1}]$, $j = 1,\dots,N+3$, $(\tau_{N+4} := \tau_1)$ $\forall\, \tau \in T$ that $X_\varphi(e^{i\varphi},\tau) \in \Gamma_j - P_j \equiv \mathit{Span}(P_{j+1} - P_j)$ and also $X_{\tau_l}(e^{i\varphi},\tau) \in \mathit{Span}(P_{j+1} - P_j)$ for $l = 1,\dots,N$, $\varphi \in (\tau_j,\tau_{j+1})$ and any $\tau \in T$ (see also (4.74) in [18]). Inserting this into (10.18) we obtain

$$h_r^l(\varphi,\tau) \longrightarrow 0 \equiv h_1^l(\varphi,\tau) \qquad \textit{for } r \nearrow 1, \tag{10.25}$$

pointwise for every $\varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\dots,N+3}$ and for any $\tau \in T$. Moreover we have:

$$\frac{\partial}{\partial r}\Big(\langle X_\varphi \wedge X, X_{\tau_l}\rangle\Big)(w,\tau)$$
$$= \langle X_{\varphi r} \wedge X, X_{\tau_l}\rangle(w,\tau) + \langle X_\varphi \wedge X_r, X_{\tau_l}\rangle(w,\tau) + \langle X_\varphi \wedge X, X_{\tau_l r}\rangle(w,\tau),$$

$\forall\, w \in B$ and for any $\tau \in T$. Hence, by formulas (2.19)–(2.21), estimates (10.2)–(10.4) and again (10.23), with ρ replaced by $2\rho - 1$, we achieve

$$| \frac{\partial}{\partial r} h_r^l(\varphi,\tau) | \leq \mathit{const.}(\delta,\bar\tau,l) \sum_{k=1}^{N+3} | re^{i\varphi} - e^{i\tau_k} |^{2\rho-1}$$
$$\leq c(\delta,\bar\tau,l) \sum_{k=1}^{N+3} \Big(\Big(\frac{| \varphi - \tau_k |}{4}\Big)^{2\rho-1} + R^{2\rho-1}\Big), \tag{10.26}$$

$\forall\, r \in (0,1)$, $\varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$ and $\forall\, \tau \in B_\varepsilon(\bar\tau)$, where we fixed some $\varepsilon \in (0, \frac{\delta}{2})$ and $R \in (0, \frac{1}{8})$ arbitrarily. Now we choose some $\epsilon > 0$ arbitrarily small. One can easily see that for an arbitrary $\tau \in T$ there holds $\mathrm{dist}(\tau, \partial T) = \frac{1}{\sqrt{2}} \min_{j=1,\ldots,N+3}\{\mid \tau_{j+1} - \tau_j \mid\}$. Hence, the union $\bigcup_{k=1}^{N+3} B_s(\tau_k)$ of intervalls is disjoint for any $\tau \in B_\varepsilon(\bar\tau)$ if we choose $s \in (0, \frac{dist(\bar\tau, \partial T) - \varepsilon}{\sqrt{2}})$, and we can estimate by (10.24) for those s:

$$\begin{aligned}
&\int_{\bigcup_{k=1}^{N+3} B_s(\tau_k)} \mid h_r^l(\varphi,\tau) \mid \, r\, d\varphi \\
&\le c(\delta,\bar\tau,l)\,(N+3) \sum_{k=1}^{N+3} \int_{B_s(\tau_k)} \Big(\frac{\mid \varphi - \tau_k \mid}{4}\Big)^\rho + R^\rho\, d\varphi, \\
&= c(\delta,\bar\tau,l)\,(N+3)^2\, 2\,\Big(\int_0^s \Big(\frac{\varphi}{4}\Big)^\rho d\varphi + s\, R^\rho\Big) \\
&= c(\delta,\bar\tau,l)\, 2\,(N+3)^2 \Big(\frac{s^{\rho+1}}{4^\rho(\rho+1)} + s\, R^\rho\Big),
\end{aligned}$$

$\forall\, r \in (0,1)$ and uniformly $\forall\, \tau \in B_\varepsilon(\bar\tau)$. Thus due to $\rho + 1 > 0$ we achieve the existence of some $\bar s(\epsilon) \in (0, \frac{dist(\bar\tau, \partial T) - \epsilon}{\sqrt{2}})$ such that

$$\int_{\bigcup_{k=1}^{N+3} B_{\bar s(\epsilon)}(\tau_k)} \mid h_r^l(\varphi,\tau) \mid \, r\, d\varphi < \frac{\epsilon}{2}, \tag{10.27}$$

$\forall\, r \in (0,1)$ and uniformly $\forall\, \tau \in B_\varepsilon(\bar\tau)$. Moreover we obtain by the mean value theorem, $h_1^l(\varphi,\tau) \equiv 0$, for every $\varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$ and any $\tau \in T$ by (10.25), and (10.26):

$$\begin{aligned}
&\int_{[0,2\pi]\setminus \bigcup_{k=1}^{N+3} B_{\bar s(\epsilon)}(\tau_k)} \mid h_r^l(\varphi,\tau) \mid \, r\, d\varphi \\
&= \int_{[0,2\pi]\setminus \bigcup_{k=1}^{N+3} B_{\bar s(\epsilon)}(\tau_k)} \mid h_1^l(\varphi,\tau) - h_r^l(\varphi,\tau) \mid \, r\, d\varphi \\
&< 2\pi\, const.(\delta,\bar\tau,l)\,(N+3)\Big(\Big(\frac{\bar s(\epsilon)}{4}\Big)^{2\rho-1} + R^{2\rho-1}\Big)\,(r - r^2)
\end{aligned}$$

$\forall\, r \in (0,1)$ and uniformly $\forall\, \tau \in B_\varepsilon(\bar\tau)$. Thus there exists some radius $\bar r < 1$ (near 1) depending on $\bar s(\epsilon)$, i.e. on ϵ, such that

$$\int_{[0,2\pi]\setminus \bigcup_{k=1}^{N+3} B_{\bar s(\epsilon)}(\tau_k)} \mid h_r^l(\varphi,\tau) \mid \, r\, d\varphi < \frac{\epsilon}{2} \tag{10.28}$$

$\forall\, r \in (\bar r(\epsilon), 1)$ and uniformly $\forall\, \tau \in B_\varepsilon(\bar\tau)$. Hence, combining (10.27) and (10.28) we achieve for any $\epsilon > 0$ the existence of some radius $\bar r(\epsilon) < 1$ such that

$$\int_0^{2\pi} \mid h_r^l(\varphi,\tau) \mid \, r\, d\varphi < \epsilon$$

for any $r \in (\bar{r}(\epsilon), 1)$ and uniformly $\forall\, \tau \in B_\varepsilon(\bar{\tau})$, which implies that

$$H_r^l := \int_0^{2\pi} h_r^l(\varphi, \cdot\,)\, r\, d\varphi \longrightarrow 0 \qquad in\ C^0(B_\varepsilon(\bar{\tau})), \tag{10.29}$$

for $r \nearrow 1$. Hence, since $\bar{\tau}$ was arbitrarily chosen in T we can conclude for any compactly contained subdomain $T' \subset\subset T$ with $Z \subset T'$ that

$$H_r^l \longrightarrow 0 \qquad in\ C^0(\bar{T}'), \tag{10.30}$$

for $r \nearrow 1$ and for any $l = 1, \ldots, N$. Now noting that $\frac{\partial}{\partial t} X^t = \sum_{l=1}^N \frac{d\tilde{\tau}_l}{dt} X^t_{\tau_l}$ and $\mid \frac{d\tilde{\tau}}{dt} \mid \leq$ *const.* on $[0, 2\pi]$ we infer immediately from (10.30):

$$\int_{\partial B_r(0)} \Big\langle X^t_\varphi \wedge X^t, \frac{\partial}{\partial t} X^t \Big\rangle\, ds = \sum_{l=1}^N \frac{d\tilde{\tau}_l}{dt}\, H_r^l(\tilde{\tau}(t)) \longrightarrow 0 \qquad in\ C^0([0, 2\pi]),$$

for $r \nearrow 1$.

$\diamond$

Moreover we are going to use the integral identity (1.9) in [13] due to Heinz (see Lemma 3.3 in [22] for a proof):

Lemma 10.1 *For Y^1, $Y^2 \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ there holds the formula*

$$\mathcal{F}_{B_r(0)}(Y^1 + Y^2) - \mathcal{F}_{B_r(0)}(Y^1) = 3 \int_{B_r(0)} \langle Y^1_u \wedge Y^1_v, Y^2 \rangle\, dw \tag{10.31}$$
$$+ \int_{B_r(0)} \langle 3\, Y^1 + Y^2, Y^2_u \wedge Y^2_v \rangle\, dw + \frac{1}{r} \int_{\partial B_r(0)} \langle Y^1, Y^2 \wedge (Y^1_\varphi - Y^2_\varphi) \rangle\, ds,$$

for a.e. $r \in (0, 1)$.

Using $\frac{\partial}{\partial t} X^t = \sum_{l=1}^N \frac{d\tilde{\tau}_l}{dt} X^t_{\tau_l}$ we may infer from (10.8), (10.9), Lebesgue's convergence theorem and the analyticity of $\tilde{\tau}$ that the functions

$$\Phi_r(t) := \int_{B_r(0)} \Big\langle X^t_u \wedge X^t_v, \frac{\partial}{\partial t} X^t \Big\rangle\, dw \tag{10.32}$$

are continuous in $t \in [0, 2\pi]/(0 \sim 2\pi)$ for any $r \in (0, 1)$ (compare with (10.16) for $r = 1$). Together with Theorem 2.1 (i), (iv), (vi) and Lemma 10.1 we can prove the following connection between these integrals Φ_r and $\mathcal{F}_{B_r(0)}(X^{(\cdot)})$:

Proposition 10.3 *There holds for a.e. $r \in (0, 1)$ the formula*

$$\frac{d}{dt} \mathcal{F}_{B_r(0)}(X^t) = 3\, \Phi_r(t) + \frac{1}{r} \int_{\partial B_r(0)} \Big\langle X^t_\varphi \wedge X^t, \frac{\partial}{\partial t} X^t \Big\rangle\, ds \tag{10.33}$$

$\forall\, t \in [0, 2\pi]/(0 \sim 2\pi)$.

Proof: Without loss of generality we may only consider some arbitrary point $t^* \in (0, 2\pi)$. We obtain by (10.31) applied to $Y^1 := X^{t^*}$ and $Y^2 := X^t - X^{t^*}$ for an arbitrarily chosen $r \in (0,1)$ which (10.31) holds for:

$$\frac{\mathcal{F}_{B_r}(X^t) - \mathcal{F}_{B_r}(X^{t^*})}{t - t^*} = 3 \int_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \right\rangle dw$$
$$+ \int_{B_r} \left\langle 2X^{t^*} + X^t, \frac{(X^t - X^{t^*})_u}{t - t^*} \wedge (X^t - X^{t^*})_v \right\rangle dw$$
$$+ \frac{1}{r} \int_{\partial B_r} \left\langle X^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \wedge (2X_\varphi^{t^*} - X_\varphi^t) \right\rangle ds, \qquad (10.34)$$

with $B_r := B_r(0)$. Firstly we consider the first integral on the right hand side. By Theorem 2.1 (vi) and by the analyticity of $\tilde{\tau}$ we know that

$$\frac{X^t(w) - X^{t^*}(w)}{t - t^*} \longrightarrow \frac{\partial}{\partial t} X^{t^*}(w) \qquad \text{pointwise } \forall\, w \in B \qquad (10.35)$$

and $t \to t^*$. Now combining (10.8) for some $\sigma < \frac{1-r}{2}$ with the analyticity of $\tilde{\tau}$ and applying Cauchy's estimates to the harmonic functions $\frac{\partial}{\partial t} X^t - \frac{\partial}{\partial t} X^{t^*}$ we achieve:

$$\frac{\partial}{\partial t} X^t \longrightarrow \frac{\partial}{\partial t} X^{t^*} \qquad \text{in } C^1(\overline{B_r(0)}) \qquad (10.36)$$

for $t \to t^*$, which implies in particular together with the mean value theorem:

$$\parallel \frac{X^t - X^{t*}}{t - t^*} \parallel_{C^0(\overline{B_r(0)})} \leq \sup_{(t^*-h, t^*+h)} \parallel \frac{\partial}{\partial t} X^t \parallel_{C^0(\overline{B_r(0)})} \leq const.(r,h), \quad (10.37)$$

for some sufficiently small chosen $h > 0$ and $\mid t - t^* \mid < h$. Hence, recalling (10.35) we infer by Lebesgue's convergence theorem:

$$\int_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \right\rangle dw \longrightarrow \int_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{\partial}{\partial t} X^{t^*} \right\rangle dw = \Phi_r(t^*), \qquad (10.38)$$

for $t \to t^*$. Now we examine the second integral in (10.34). Using that $X^t \equiv \tilde{\psi}(\tilde{\tau}(t)) = \psi(\tilde{\tau}(t))$ due to $Z \subset K(f)$ and Corollary 2.4 we have by Theorem 2.1 (i) and the analyticity of $\tilde{\tau}$ that $X^t \longrightarrow X^{t^*}$ in $C^0(\bar{B})$ for $t \to t^*$. Thus together with Cauchy's estimates applied to $X^t - X^{t^*}$ we achieve:

$$X^t \longrightarrow X^{t^*} \qquad \text{in } C^1(\overline{B_r(0)}) \qquad (10.39)$$

for $t \to t^*$. Moreover we infer from (10.36) together with the mean value theorem:

$$\parallel \frac{X_u^t - X_u^{t*}}{t - t^*} \parallel_{C^0(\overline{B_r(0)})} \leq \sup_{(t^*-h, t^*+h)} \parallel \frac{\partial}{\partial t} X_u^t \parallel_{C^0(\overline{B_r(0)})} \leq const.(r,h), \qquad (10.40)$$

for some sufficiently small chosen $h > 0$ and $| t - t^* | < h$. Hence, together with (10.39) we achieve by Lebesgue's convergence theorem:

$$\int_{B_r} \left\langle 2X^{t^*} + X^t, \frac{(X^t - X^{t^*})_u}{t - t^*} \wedge (X^t - X^{t^*})_v \right\rangle dw \longrightarrow 0 \tag{10.41}$$

for $t \to t^*$. Finally we examine the third integral in (10.34). We deduce from (10.39) especially:

$$X^t_\varphi \longrightarrow X^{t^*}_\varphi \qquad in\ C^0(\partial B_r(0)),$$

for $t \to t^*$. Thus together with (10.35) and (10.37) we infer again by Lebesgue's convergence theorem:

$$\int_{\partial B_r} \left\langle X^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \wedge (2X^{t^*}_\varphi - X^t_\varphi) \right\rangle ds \longrightarrow \int_{\partial B_r} \left\langle X^{t^*}, \frac{\partial}{\partial t} X^{t^*} \wedge X^{t^*}_\varphi \right\rangle ds$$
$$= \int_{\partial B_r} \left\langle X^{t^*}_\varphi \wedge X^{t^*}, \frac{\partial}{\partial t} X^{t^*} \right\rangle ds,$$

for $t \to t^*$. Now combining this with (10.34), (10.38) and (10.41) we see indeed that $\lim_{t \to t^*} \frac{\mathcal{F}_{B_r}(X^t) - \mathcal{F}_{B_r}(X^{t^*})}{t - t^*}$ exists and coincides with the right hand side of (10.33) for any $t^* \in (0, 2\pi)$, thus $\forall\, t \in [0, 2\pi]/(0 \sim 2\pi)$, and for a.e. $r \in (0, 1)$.

$\diamond$

In the sequel we have to examine some notions considered in [34] in order to use Sauvigny's result, Satz 2 in [34], correctly:

Definition 10.1 *Let $X \in \tilde{\mathcal{M}}(\Gamma)$ be a fixed minimal surface without any branch points. For any map $Y \in C^0(B, \mathbb{R}^3)$ we consider its normal component w. r. to X:*

$$Y^* := \langle Y, \xi \rangle\, \xi = Y - \frac{1}{E}(\langle Y, X_u \rangle\, X_u + \langle Y, X_v \rangle\, X_v) \qquad \text{on } B,$$

where ξ denotes the unit normal field of X, as defined in (2.34).

Furthermore we have to compare our definition of the quadratic form J^X assigned to a minimal surface X with the following one, considered in [34]:

Definition 10.2 *Let $X \in \tilde{\mathcal{M}}(\Gamma)$ be a fixed minimal surface without branch points. For any $\phi \in C^1(B, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ we define:*

$$I^X(\phi) := \int_B |\, (\phi_u)^* \,|^2 + |\, (\phi_v)^* \,|^2 + \frac{2}{E}(\langle \phi_u, X_u \rangle \langle \phi_v, X_v \rangle - \langle \phi_u, X_v \rangle \langle \phi_v, X_u \rangle)\, dw.$$

Definition 10.3 *We term the normal space of a minimal surface $X \in \tilde{\mathcal{M}}(\Gamma)$ without branch points:*

$$\mathcal{N}_X := \{\phi \in C^1(B,\mathbb{R}^3) \cap C^0(\bar{B},\mathbb{R}^3) \cap \mathring{H}^{1,2}(B,\mathbb{R}^3) \mid \phi \mid\mid \xi \quad \text{on } \bar{B}\}.$$

As pointed out on p. 173 in [34] we are going to prove now

Lemma 10.2 *Let $X \in \tilde{\mathcal{M}}(\Gamma)$ be a minimal surface without branch points. For any $\phi \in \mathcal{N}_X$ there holds $\varphi := \langle \phi, \xi \rangle \in \mathring{H}^{1,2}(B) \cap C^0(\bar{B})$ and*

$$I^X(\phi) = J^X(\varphi). \tag{10.42}$$

Proof: We have $\phi = \phi^* = \langle \phi, \xi \rangle \, \xi$ and therefore:

$$\phi_u = \langle \phi, \xi \rangle_u \, \xi + \langle \phi, \xi \rangle \, \xi_u \qquad \text{on } B. \tag{10.43}$$

Thus recalling the fundamental equations (see Lemma 1 in [31])

$$\begin{aligned} \xi_u \wedge \xi_v = K\, X_u \wedge X_v = K\, E\, \xi \qquad &\text{on } B, \\ \mid \xi_u \mid = \mid \xi_v \mid, \quad \langle \xi_u, \xi_v \rangle = 0 \qquad &\text{on } B, \end{aligned}$$

we obtain $\langle \xi_u, \xi \rangle \equiv 0 \equiv \langle \xi_v, \xi \rangle$ on B in particular and conclude that

$$(\phi_u)^* = \langle \phi, \xi \rangle_u \, \xi \qquad \text{on } B.$$

Hence, we achieve $\mid (\phi_u)^* \mid^2 = \mid \langle \phi, \xi \rangle_u \mid^2 \equiv \mid \varphi_u \mid^2$ and analogously $\mid (\phi_v)^* \mid^2 = \mid \varphi_v \mid^2$, yielding

$$\mid (\phi_u)^* \mid^2 + \mid (\phi_v)^* \mid^2 = \mid \nabla \varphi \mid^2 \qquad \text{on } B, \tag{10.44}$$

which implies especially $\varphi \in \mathring{H}^{1,2}(B) \cap C^0(\bar{B})$ by $\phi \in \mathcal{N}_X$ and $\xi \in C^0(\bar{B},\mathbb{R}^3) \cap C^1(B,\mathbb{R}^3)$. Moreover we compute by (10.43):

$$\langle \phi_u, X_u \rangle = \langle \phi, \xi \rangle \, \langle \xi_u, X_u \rangle \equiv -\varphi\, L \qquad \text{on } B,$$

and analogously we obtain

$$\begin{aligned} \langle \phi_v, X_v \rangle = \langle \phi, \xi \rangle \, \langle \xi_v, X_v \rangle &\equiv -\varphi\, N, \\ \langle \phi_u, X_v \rangle = \langle \phi, \xi \rangle \, \langle \xi_u, X_v \rangle &\equiv -\varphi\, M = \langle \phi_v, X_u \rangle, \end{aligned}$$

where we used the notation in [6], p. 17. Therefore we arrive at:

$$\begin{aligned} &\frac{2}{E}(\langle \phi_u, X_u \rangle \, \langle \phi_v, X_v \rangle - \langle \phi_u, X_v \rangle \, \langle \phi_v, X_u \rangle) \\ &\quad = \frac{2}{E}(L\, N - M^2)\, \varphi^2 = 2\, K E\, \varphi^2, \end{aligned} \tag{10.45}$$

(see p. 19 in [6]). Thus together with (10.44) we achieve (10.42).

◇

Now by Satz 1 in [34] we can derive as in Satz 2 in [34]:

Theorem 10.1 *Let $\bar{\tau}$ be some arbitrary point of Z, then $D^2\tilde{f}(\bar{\tau})$ is positive semidefinite.*

Proof: As in [34], pp. 174–182, we consider for an arbitrarily fixed vector $\alpha \in \mathbb{R}^N$ the following family of harmonic surfaces:

$$Y(\,\cdot\,,\epsilon) := X(\,\cdot\,,\bar{\tau}+\epsilon\alpha) \qquad \textit{on } \bar{B}, \quad \textit{for } \epsilon \in (-\epsilon_0,\epsilon_0),$$

where $\epsilon_0 > 0$ is chosen sufficiently small. By Satz 1 in [34] we know that

$$\phi := \Big(\frac{\partial}{\partial\epsilon} Y(\,\cdot\,,\epsilon)\,|_{\epsilon=0}\Big)^* \in \mathcal{N}_{X(\,\cdot\,,\bar{\tau})}$$

and that there holds

$$\frac{d^2}{d\epsilon^2}\tilde{f}(\bar{\tau}+\epsilon\alpha)\,|_{\epsilon=0} \equiv \frac{d^2}{d\epsilon^2}\mathcal{D}(Y(\,\cdot\,,\epsilon))\,|_{\epsilon=0} \geq 2\,I^{X(\,\cdot\,,\bar{\tau})}(\phi). \tag{10.46}$$

Moreover we know that $J^{\bar{\tau}} \geq 0$ on $\mathring{H}^{1,2}(B)$ by definition of Z and Lemma 3.1. We may apply this to the function $\varphi := \langle \phi, \xi\rangle$ since Lemma 10.2 guarantees that $\langle\phi,\xi\rangle \in \mathring{H}^{1,2}(B) \cap C^0(\bar{B})$. Hence, we conclude by (10.42) and (10.46):

$$\langle \alpha, D^2\tilde{f}(\bar{\tau})\,\alpha\rangle = \frac{d^2}{d\epsilon^2}\tilde{f}(\bar{\tau}+\epsilon\alpha)\,|_{\epsilon=0} \geq 2\,J^{\bar{\tau}}(\varphi) \geq 0,$$

for any $\alpha \in \mathbb{R}^N$, which proves the assertion of the theorem.

$\diamond$

Now we are able to prove

Theorem 10.2 *There holds $\mathcal{F}(X^{(\,\cdot\,)}) \in C^1([0,2\pi])$ with*

$$\frac{d}{dt}\mathcal{F}(X^t) = 3\,\Phi_1(t) \qquad \text{for } t \in [0,2\pi]. \tag{10.47}$$

In particular, $\mathcal{F}(X^{(\,\cdot\,)})$ is strictly monotonic on $[0,2\pi]/(0 \sim 2\pi)$.

Proof: As $\mathrm{trace}(\tilde{\tau}) = K(\tilde{f})^1_{\tau^*}$ we have $\nabla\tilde{f}(\tilde{\tau}(t)) \equiv 0$ implying $\tilde{f}(\tilde{\tau}(t)) \equiv$ const., where we used that $K(\tilde{f})^1_{\tau^*} = Z$ is an analytic curve, and therefore:

$$\begin{aligned} 0 \equiv \frac{d^2}{dt^2}\tilde{f}(\tilde{\tau}(t)) &= \Big\langle \nabla\tilde{f}(\tilde{\tau}(t)), \frac{d^2\tilde{\tau}}{dt^2}\Big\rangle + \Big\langle \frac{d\tilde{\tau}}{dt}, D^2(\tilde{f})(\tilde{\tau}(t))\,\frac{d\tilde{\tau}}{dt}\Big\rangle \\ &= \Big\langle \frac{d\tilde{\tau}}{dt}, D^2(\tilde{f})(\tilde{\tau}(t))\,\frac{d\tilde{\tau}}{dt}\Big\rangle, \end{aligned} \tag{10.48}$$

on $[0,2\pi]/(0 \sim 2\pi)$. Moreover Theorem 10.1 yields the positive semidefiniteness of $D^2(\tilde{f})(\tilde{\tau}(t))\ \ \forall\, t \in [0,2\pi]$. Together with the symmetry of $D^2(\tilde{f})(\tilde{\tau}(t))$ we achieve the existence of a symmetric root of $D^2(\tilde{f})(\tilde{\tau}(t))$, i.e. there exists some symmetric matrix $R(t)$ with $D^2(\tilde{f})(\tilde{\tau}(t)) = R(t)\cdot R(t)$, which yields together with (10.48):

$$0 \equiv \Big\langle \frac{d\tilde{\tau}}{dt}, D^2(\tilde{f})(\tilde{\tau}(t))\,\frac{d\tilde{\tau}}{dt}\Big\rangle = \Big\langle \frac{d\tilde{\tau}}{dt}, R(t)^\top \cdot R(t)\,\frac{d\tilde{\tau}}{dt}\Big\rangle = \mid R(t)\,\frac{d\tilde{\tau}}{dt}\mid^2,$$

$\forall\, t \in [0,2\pi]$. Hence, we arrive at

$$D^2(\tilde{f})(\tilde{\tau}(t))\,\frac{d\tilde{\tau}}{dt} = R(t)\cdot R(t)\,\frac{d\tilde{\tau}}{dt} \equiv R(t)\,0 = 0 \qquad \forall\, t \in [0,2\pi].$$

Thus on account of Satz 1 in [18] we can conclude that

$$\Big\langle \xi^t, \sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k}\Big\rangle \in Ker(A^{\tilde{\tau}(t)}), \tag{10.49}$$

with $\xi^t := \frac{X^t_u \wedge X^t_v}{|X^t_u \wedge X^t_v|}$. Now due to $\kappa(\tilde{\tau}(\,\cdot\,)) \equiv 0$ by definition of Z we infer from Corollary 2.2 that the functions $\{X^t_{\tau_l}\}_{l\in\{1,\ldots,N\}}$ are linearly independent on B for any $t \in [0,2\pi]$. Hence, by the regularity of the parametrization $\tilde{\tau}$ of Z, i.e. by $\frac{d\tilde{\tau}}{dt} \neq 0$ on $[0,2\pi]$, we gain that

$$\sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k} \not\equiv 0 \qquad \textit{on } \ B, \tag{10.50}$$

$\forall\, t \in [0,2\pi]$. Now on account of Satz 1 in [18] Heinz assigns to any $\tau \in K(\tilde{f})$ the linear map

$$C^\tau : V^\tau := \{Y = \sum_{k=1}^{N} \alpha_k X_{\tau_k}(\,\cdot\,,\tau) \mid \alpha \in Ker(D^2(\tilde{f})(\tau))\} \longrightarrow Ker(A^\tau),$$

defined by $Y \mapsto \langle \xi(\tau), Y\rangle$. By (5.7), (5.7') and (5.17) in [18] we know the formula

$$\dim\ \mathrm{Ker}(C^\tau) = 2\kappa(\tau) - \sharp\{\text{boundary branch points of } X(\,\cdot\,,\tau)\},$$

for any $\tau \in K(\tilde{f})$. Now by $\kappa(\tilde{\tau}(t)) = 0$ we infer $\dim\ \mathrm{Ker}(C^{\tilde{\tau}(t)}) = 0$, i.e. that $C^{\tilde{\tau}(t)}$ is injective $\forall\, t \in [0,2\pi]$, which implies by (10.49), (10.50) and $\lambda_{\min}(A^{\tilde{\tau}(t)}) = 0$:

$$\Big\langle \xi^t, \frac{\partial}{\partial t} X^t\Big\rangle = \Big\langle \xi^t, \sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k}\Big\rangle \in \mathrm{ES}_{(\lambda_{\min}=0)}(A^{\tilde{\tau}(t)}) \setminus \{0\}$$

$\forall\, t \in [0, 2\pi]$. Hence, we infer from Theorem 8.2 (ii) and $\mid X_u^t \wedge X_v^t \mid = \frac{1}{2} \mid DX^t \mid^2 > 0$ on B:

$$\mid \left\langle X_u^t \wedge X_v^t, \frac{\partial}{\partial t} X^t \right\rangle \mid > 0 \qquad \text{on } B, \qquad \forall\, t \in [0, 2\pi]. \tag{10.51}$$

Now we choose some arbitrary sequence of radii $r_n \nearrow 1$ such that formula (10.33) holds for each r_n and conclude together with (10.32) that

$$\Phi_{r_n}(t) \leq \Phi_{r_{n+1}}(t) \qquad \text{or} \qquad \Phi_{r_n}(t) \geq \Phi_{r_{n+1}}(t) \qquad \forall\, t \in [0, 2\pi], \tag{10.52}$$

$\forall\, n \in \mathbb{N}$. Lebesgue's convergence theorem guarantees that $\Phi_{r_n}(t) \to \Phi_1(t)$ pointwise for any $t \in [0, 2\pi]$. Since we know by (10.16) and (10.32) that the functions Φ_r are continuous on $[0, 2\pi]$ for any $r \in (0, 1]$ we can apply Dini's theorem to the monotonic sequence in (10.52) yielding

$$\Phi_{r_n} \longrightarrow \Phi_1 \qquad \text{in } C^0([0, 2\pi]), \tag{10.53}$$

for $n \to \infty$. If we insert also the convergence (10.17) into formula (10.33) we obtain together with (10.53):

$$\frac{d}{dt} \mathcal{F}_{B_{r_n}}(X^{(\,\cdot\,)}) \longrightarrow 3\, \Phi_1 \qquad \text{in } C^0([0, 2\pi]), \tag{10.54}$$

for $n \to \infty$, which especially implies the equicontinuity of $\{\mathcal{F}_{B_{r_n}}(X^{(\,\cdot\,)})\}$ on $[0, 2\pi]$. Furthermore due to the existence of $\mathcal{F}_B(X^t)$ by estimate (10.2) we can infer from Lebesgue's convergence theorem:

$$\mathcal{F}_{B_{r_n}}(X^t) \longrightarrow \mathcal{F}_B(X^t) \qquad \text{pointwise } \forall\, t \in [0, 2\pi]$$

and for $n \to \infty$. Hence, combining this with the proof of Arzela-Ascoli's theorem and (10.54) we obtain:

$$\mathcal{F}_{B_{r_n}}(X^{(\,\cdot\,)}) \longrightarrow \mathcal{F}_B(X^{(\,\cdot\,)}) \qquad \text{in } C^1([0, 2\pi]),$$

for $n \to \infty$, which shows indeed by (10.54):

$$\frac{d}{dt} \mathcal{F}_B(X^t) = \lim_{n \to \infty} \frac{d}{dt} \mathcal{F}_{B_{r_n}}(X^t) = 3\, \Phi_1(t) \qquad \forall\, t \in [0, 2\pi].$$

Now together with (10.51) and (10.16) we can conclude that $\frac{d}{dt} \mathcal{F}_B(X^{(\,\cdot\,)}) > 0$ or < 0 on $[0, 2\pi]$, i.e. that $\mathcal{F}_B(X^{(\,\cdot\,)})$ is strictly monotonic on $[0, 2\pi]$.

$\diamond$

Since we trivially have $\mathcal{F}_B(X^0) = \mathcal{F}_B(X^{2\pi})$ by $\tilde{\tau}(0) = \tilde{\tau}(2\pi)$ in contradiction to the strict monotonicity of $\mathcal{F}_B(X^{(\,\cdot\,)})$ on $[0, 2\pi]$ we finally proved Theorem 1.1.

Now together with the compactness of the set $(\mathcal{M}_s(\Gamma), \|\cdot\|_{C^0})$, on account of Theorems 3.1 and 4.1, we immediately infer its finiteness.

Chapter 11

Local boundedness of $\sharp(\mathcal{M}_s(\Gamma))$

In this chapter we shall extend the result of Theorem 1.1 to the "local boundedness" of $\sharp(\mathcal{M}_s(\Gamma))$, i.e. precisely to Theorem 1.2. To this end we also have to extend our notation. Firstly we note that we have a correspondence between "admissible" vertex tuples $P := (P_1, ..., P_{N+3}) \in \mathbb{R}^{3N+9}$ (see (1.3)) and their resulting polygons $\Gamma(P)$. For any such fixed P we shall consider now the maps $\psi(P, \cdot\,)$ and $\tilde{\psi}(P, \cdot\,)$ as defined in (2.3) and (2.4) for the polygon $\Gamma(P)$, resp. $f(P, \cdot\,) := \mathcal{D}(\psi(P, \cdot\,))$ and $\tilde{f}(P, \cdot\,) := \mathcal{D}(\tilde{\psi}(P, \cdot\,))$, where we will use the notation $X(\,\cdot\,, P, \tau)$ for $\tilde{\psi}(P, \tau)$ again. Moreover we will use the abbreviations $A^{(P,\tau)} := A^{X(\,\cdot\,,P,\tau)}$ for the Schwarz operator and $J^{(P,\tau)} := J^{X(\,\cdot\,,P,\tau)}$ for the quadratic form assigned to some minimal surface $X(\,\cdot\,, P, \tau)$. On account of point (v) of Theorem 2.1 we can introduce the notation

$$K_s(f(P, \cdot\,)) := (\psi(P, \cdot\,))^{-1}(\mathcal{M}_s(\Gamma(P))) \tag{11.1}$$

for the set of critical points of $f(P, \cdot\,)$ that correspond to the immersed stable minimal surfaces in $\mathcal{C}^*(\Gamma(P))$.

Remark 11.1 *We shall remark here that $\mathcal{M}_s(\Gamma)$ is nonvoid for any extreme simple closed polygon Γ. To see this we shall consider a global minimizer $\bar{X}$ of $\mathcal{A}$ on $\mathcal{C}^*(\Gamma)$, i.e.*

$$\mathcal{A}(\bar{X}) = \inf_{\mathcal{C}^*(\Gamma)} \mathcal{A}, \tag{11.2}$$

whose existence, harmonicity and conformality are guaranteed by Theorems 2 and 3 in [21]. In fact Alt's papers [2], [3] yield that such a minimal surface is free of branch points on B, and $\bar{X}$ does not have any boundary branch points neither on account of the extremeness of Γ and Theorem 4.1, hence $\bar{X}$ is immersed. Moreover $\bar{X}$ is stable due to $\bar{X} + \epsilon\varphi\bar{\xi} \in \mathcal{C}^*(\Gamma)$, for any $\varphi \in C_c^\infty(B)$ and any $\epsilon \in \mathbb{R}$, and thus $J^X(\varphi) = \frac{d^2}{d\epsilon^2}\mathcal{A}(\bar{X} + \epsilon\varphi\,\bar{\xi})\,|_{\epsilon=0} \geq 0$ by (11.2), where we used that the normal $\bar{\xi}$ of $\bar{X}$ is smooth on B.

Now we fix some extreme simple closed polygon $\Gamma^* \subset \mathbb{R}^3$ with vertex tuple P^* which meets the requirement that the angles at its vertices are different from 0, $\frac{\pi}{2}$ and π. Following Tomi's ideas of the proof of Lemma 4 in [36] we shall prove analogously:

Lemma 11.1 *Let $U \subset\subset T$ be some arbitrary open neighborhood of the (nonvoid) finite set $K_s(f(P^*, \cdot))$. Then there exists some neighborhood O of P^* in $\mathbb{R}^{3N+9}$ such that there holds*

$$K_s(f(P, \cdot)) \subset U \qquad \forall\, P \in O.$$

Proof: We suppose the statement was wrong. Thus there would have to exist some sequence $\{P^n\}$ in $\mathbb{R}^{3N+9}$ with

$$P^n \longrightarrow P^*, \qquad \textit{for } n \to \infty, \tag{11.3}$$

and according points $\tau^n \in K_s(f(P^n, \cdot)) \cap U^c \;\; \forall\, n \in \mathbb{N}$. Clearly there exists some converging subsequence $\{\tau^{n_j}\}$, which we rename $\{\tau^n\}$ again, and we claim the limit point τ^* to be contained in T, i.e.:

$$\tau^n \longrightarrow \tau^* \in T \cap U^c. \tag{11.4}$$

For let us assume that there would hold $\text{dist}(\tau^n, \partial T) \longrightarrow 0$. By the definition of T this assumption implies the existence of some subsequence $\{\tau^{n_k}\}$ and some consequtive indices $l_1, l_2 = l_1 + 1$ in $\{1, \ldots, N+3\}$ (where $l_1 := N+3$ if $l_2 = 1$ or $l_2 := 1$ if $l_1 = N+3$) and some angle $\varphi \in [0, \pi]/(0 \sim 2\pi)$ such that

$$\tau_{l_1}^{n_k} \longrightarrow \varphi \longleftarrow \tau_{l_2}^{n_k}.$$

Now we consider the null sequence $\delta_k := \frac{1}{8k} + \max_{j=1,2}\{\mid e^{i\tau_{l_j}^{n_k}} - e^{i\varphi} \mid\}$ which we may assume to be contained in $(0,1)$ for any $k \in \mathbb{N}$. By the Courant-Lebesgue Lemma, applied to the surfaces $X(\,\cdot\,, P^{n_k}, \tau^{n_k})$, there exists some $\rho_k \in (\delta_k, \sqrt{\delta_k})$ such that there holds

$$\mid X(w_1^k, P^{n_k}, \tau^{n_k}) - X(w_2^k, P^{n_k}, \tau^{n_k}) \mid \leq \sqrt{\frac{4\pi\, \mathcal{D}(X(\,\cdot\,, P^{n_k}, \tau^{n_k}))}{\log \frac{1}{\delta_k}}} \tag{11.5}$$

in the points $\{w_1^k, w_2^k\} := \partial B_{\rho_k}(e^{i\varphi}) \cap \partial B$ and $\forall\, k \in \mathbb{N}$. Moreover by (11.3) the corresponding polygons $\Gamma^n := \Gamma(P^n)$ satisfy a uniform chord-arc condition, i.e. there exists some constant C such that there holds for any fixed $n \in \mathbb{N}$ and any pair of points $Q_1, Q_2 \in \Gamma^n$:

$$\mathcal{L}(\Gamma^n \mid_{(Q_1,Q_2)}) \leq C \mid Q_1 - Q_2 \mid, \tag{11.6}$$

where $\mathcal{L}(\Gamma^n \mid_{(Q_1,Q_2)})$ denotes the length of the shorter subarc $\Gamma^n \mid_{(Q_1,Q_2)}$ on Γ^n connecting Q_1 and Q_2. Furthermore we know that there exists some constant L such that $\mathcal{L}(\Gamma^n) \leq L \ \forall n \in \mathbb{N}$ again due to (11.3). Finally, since the τ^n are assumed to be critical points of $f(P^n, \cdot)$ implying that $X(\,\cdot\,, P^n, \tau^n) \equiv \tilde{\psi}(P^n, \tau^n)$ coincide with $\psi(P^n, \tau^n)$ by Corollary 2.4 and thus are elements of $\mathcal{M}(\Gamma^n)$ the isoperimetric inequality yields:

$$\mathcal{D}(X(\,\cdot\,, P^n, \tau^n)) = \mathcal{A}(X(\,\cdot\,, P^n, \tau^n)) \leq \frac{1}{4\pi} L^2 \qquad \forall\, n \in \mathbb{N}. \tag{11.7}$$

Now combining this with (11.5) and (11.6), applied to $Q_j^k := X(w_j^k, P^{n_k}, \tau^{n_k})$ for $j = 1, 2$, we arrive at:

$$\mathcal{L}(\Gamma^{n_k} \mid_{(Q_1^k, Q_2^k)}) \leq \frac{C\, L}{\sqrt{\log \frac{1}{\delta_k}}} \longrightarrow 0 \qquad \textit{for } k \to \infty. \tag{11.8}$$

Together with (11.3) this shows in particular that $\Gamma^{n_k} \mid_{(Q_1^k, Q_2^k)}$ can contain at most one of the points $\{P^{n_k}_{N+l}\}_{l=1,2,3}$ of the respective three point condition for sufficiently large k. Moreover $B_{\rho_k}(e^{i\varphi}) \cap \partial B$ can contain at most one of the three points $\{e^{i\pi}, e^{i\frac{3}{2}\pi}, 1\}$ of the three point condition for sufficiently large k on account of $\rho_k < \sqrt{\delta_k} \to 0$. Together with the weak monotonicity and the three point condition imposed on the boundary values $X(\,\cdot\,, P^{n_k}, \tau^{n_k}) \mid_{\partial B}$ this guarantees that

$$\textit{trace}(X(\,\cdot\,, P^{n_k}, \tau^{n_k}) \mid_{B_{\rho_k}(e^{i\varphi}) \cap \partial B}) = \Gamma^{n_k} \mid_{(Q_1^k, Q_2^k)} \qquad \forall\, k > K, \tag{11.9}$$

for some large $K \in \mathbb{N}$. Moreover there holds $e^{i\tau^{n_k}_{l_j}} \in B_{\rho_k}(e^{i\varphi}) \cap \partial B$, for any k and $j = 1, 2$, by $\rho_k > \delta_k$ and the definition of δ_k. Thus we can infer from (2.2) and (11.9) that $P^{n_k}_{l_j} = X(e^{i\tau^{n_k}_{l_j}}, P^{n_k}, \tau^{n_k}) \in \Gamma^{n_k} \mid_{(Q_1^k, Q_2^k)}$, for $j = 1, 2$ and $k > K$, and thus together with (11.8):

$$\mid P^{n_k}_{l_1} - P^{n_k}_{l_2} \mid \leq \mathcal{L}(\Gamma^{n_k} \mid_{(Q_1^k, Q_2^k)}) \longrightarrow 0 \qquad \textit{for } k \to \infty,$$

which contradicts $\mid P^{n_k}_{l_1} - P^{n_k}_{l_2} \mid \to \mid P^*_{l_1} - P^*_{l_2} \mid > 0$ by (11.3). Thus we proved in fact (11.4). Now we know by Satz 2 in [16] that $X(\,\cdot\,, \cdot\,, \cdot\,)$ behaves analytically about a point (w, P^*, τ^*) in $B \times \mathbb{R}^{3N+9} \times T$ for any fixed $w \in B$, where we use the additional requirement on P^* that the angles at the vertices P^*_j, $j = 1, \ldots, N+3$, are different from $\frac{\pi}{2}$ (Requirement (H) in [16]). Thus we conclude in particular by (11.3) and (11.4) that

$$X(w, P^{n_k}, \tau^{n_k}) \longrightarrow X(w, P^*, \tau^*) \qquad \textit{for } k \to \infty, \tag{11.10}$$

pointwise for any $w \in B$. Moreover (11.3) implies the Frechét convergence of $\{\Gamma^n\}$ to Γ^*. Thus together with $X(\,\cdot\,, P^{n_k}, \tau^{n_k}) \in \mathcal{M}(\Gamma^{n_k})$ and (11.7) we see that all requirements of a compactness result for boundary values due to Nitsche (see [29], p. 208) are fulfilled which implies the existence of some further subsequence of $\{X(\,\cdot\,, P^{n_k}, \tau^{n_k})\}$ (to be renamed again) that satisfies:

$$X(\,\cdot\,, P^{n_k}, \tau^{n_k})\,|_{\partial B} \longrightarrow \gamma \qquad in\ C^0(\partial B, \mathbb{R}^3), \tag{11.11}$$

for some continuous, weakly monotonic map $\gamma : \partial B \longrightarrow \Gamma^*$ onto Γ^* satisfying the appropriate three point condition $\gamma(e^{i\tau_{N+l}}) = P^*_{N+l}$, for $l = 1, 2, 3$. Now let $\bar{X}$ denote the unique harmonic extension of γ onto $\bar{B}$, thus being contained in $\mathcal{C}^*(\Gamma^*)$. Due to (11.11) we infer from the maximum principle for harmonic functions and Cauchy's estimates:

$$X(\,\cdot\,, P^{n_k}, \tau^{n_k}) \longrightarrow \bar{X} \qquad in\ C^0(\bar{B}, \mathbb{R}^3) \quad and \quad C^1_{loc}(B, \mathbb{R}^3) \tag{11.12}$$

for $k \to \infty$. In combination with (11.10) we obtain especially that $\bar{X}$ and $X(\,\cdot\,, P^*, \tau^*)$ coincide on B and thus on $\bar{B}$ and that they are contained in $\mathcal{M}(\Gamma^*)$ on account of the properties of γ and (11.12). Now since the minimal surfaces $X(\,\cdot\,, P^{n_k}, \tau^{n_k})$ do not have any branch points on $\bar{B}$ and are stable we can apply Sauvigny's Theorem 1 in [31], i.e. Prop. 3.2, which yields that the limit surface $\bar{X}$ is free of interior branch points again. Moreover since $\bar{X}$ is bounded by the *extreme* polygon Γ^* we infer from Theorem 4.1 that $\bar{X}$ does not have any boundary branch points, thus it is an immersed minimal surface. Moreover (11.12) yields by Prop. 3.1 that $\bar{X}$ is also stable again. Hence, all together we infer that $\bar{X} = X(\,\cdot\,, P^*, \tau^*) \in \mathcal{M}_s(\Gamma^*)$. By point (iii) of Theorem 2.1 we obtain especially that $\bar{X} \equiv \tilde{\psi}(P^*, \tau^*) = \psi(P^*, \tau^*)$. Thus we arrive at $\tau^* = \psi(P^*, \cdot\,)^{-1}(\bar{X}) \in K_s(f(P^*, \cdot\,))$, in contradiction to $\tau^* \notin U$ by (11.4).

◇

11.1 Proof of Theorem 1.2

Now we can finally derive the

Proof of Theorem 1.2:
Let P^* be the vertex tuple corresponding to the polygon Γ^*. Since Γ^* is required to be an *extreme* closed polygon we infer from Theorem 1.1, Remark 11.1 and (11.1) that $K_s(f(P^*, \cdot\,))$ is a nonvoid and finite set. Now we choose one of its points τ^* arbitrarily and recall that $X(\,\cdot\,, P^*, \tau^*) \equiv \tilde{\psi}(P^*, \tau^*)$ coincides with $\psi(P^*, \tau^*)$ by Corollary 2.4 and is thus an element of $\mathcal{M}_s(\Gamma^*)$. Therefore we have $J^{(P^*,\tau^*)} \geq 0$ on $C^\infty_c(B)$ and thus on $\mathring{H}^{1,2}(B)$ by Lemma 3.1, which implies that there can arise only the following two cases for the smallest eigenvalue $\lambda_{\min}(A^{(P^*,\tau^*)})$ of $A^{(P^*,\tau^*)}$ on account of Theorem 8.1:
Case 1: $\lambda_{\min}(A^{(P^*,\tau^*)}) > 0$ or Case 2: $\lambda_{\min}(A^{(P^*,\tau^*)}) = 0$.
In Case 1 we infer that $\text{kernel}(A^{(P^*,\tau^*)}) = \{0\}$. Thus in combination with the fact that $X(\,\cdot\,, P^*, \tau^*)$ is an immersed minimal surface we can immediately

derive from formula (2.15) that

$$\operatorname{rank} D^2_\tau(\tilde{f}(P^*,\tau^*)) = N. \tag{11.13}$$

Since $\tilde{f}$ is real analytic, thus smooth in particular, about the point (P^*,τ^*) by Satz 3 in [16] there exists some neighborhood $O^* \times U^*$ of (P^*,τ^*) in $\mathbb{R}^{3N+9} \times T$ such that there holds:

$$\det D_\tau(\nabla_\tau \tilde{f}(P,\tau)) \neq 0 \qquad \forall\, (P,\tau) \in O^* \times U^*. \tag{11.14}$$

Also noting that the point (P^*,τ^*) solves the N equations $\nabla_\tau \tilde{f}(P,\tau) = 0$ the implicit function theorem in its real analytic version (see e.g. [5], p. 268) applied to $\nabla_\tau \tilde{f} \in C^\omega(O^* \times U^*, \mathbb{R}^N)$ yields:
There exist neighborhoods $U_1^* \subset U^*$ of τ^* and $O_1^* \subset O^*$ of P^*, where O_1^* has to be chosen sufficiently small depending on the choice of U_1^*, and some function $F \in C^\omega(O_1^*, U_1^*)$, such that there holds $F(P^*) = \tau^*$ and

$$\{(P,\tau) \in O_1^* \times U_1^* \mid \nabla_\tau \tilde{f}(P,\tau) = 0\} = \{(P, F(P)) \mid P \in O_1^*\} = \operatorname{graph}(F).$$

Hence, for any $P \in O_1^*$ there holds:

$$K(\tilde{f}(P,\,\cdot\,)) \cap U_1^* = \{F(P)\}$$

and thus by $K_s(f(P,\,\cdot\,)) \subset K(\tilde{f}(P,\,\cdot\,))$ on account of Corollary 2.4:

$$\sharp(K_s(f(P,\,\cdot\,)) \cap U_1^*) \leq 1. \tag{11.15}$$

In Case 2 we infer from Theorem 8.2 that

$$\dim \operatorname{Ker}(A^{(P^*,\tau^*)}) = \dim \operatorname{ER}_{\lambda_{\min}=0}(A^{(P^*,\tau^*)}) = 1,$$

which implies again by formula (2.15) that

$$\operatorname{rank} D^2_\tau(\tilde{f}(P^*,\tau^*)) = N - 1. \tag{11.16}$$

Thus on account of the symmetry of $D^2_\tau(\tilde{f})$ there exists a uniquely determined permutation of the coordinates $\tau_1, \dots, \tau_N$ in T and furthermore some neighborhood $O^* \times U^*$ of the point (P^*,τ^*) in $\mathbb{R}^{3N+9} \times T$ such that there holds:

$$\det D_{\hat{\tau}}(\nabla_{\hat{\tau}} \tilde{f}(P,(\hat{\tau},\tau_N))) \neq 0 \qquad \forall\, (P,(\hat{\tau},\tau_N)) \in O^* \times U^*, \tag{11.17}$$

where we abbreviated by $\hat{\tau} := (\tau_1, \dots, \tau_{N-1})$ the tuple of the first $N-1$ permuted coordinates in T. Now let $\hat{U} \subset \mathbb{R}^{N-1}$ and $I \subset \mathbb{R}$ be neighborhoods of $\hat{\tau}^*$ and τ_N^* respectively such that $\hat{U} \times I \subset U^*$. Also noting that $(P^*, \hat{\tau}^*, \tau_N^*)$ is a solution of the $N-1$ equations $\nabla_{\hat{\tau}} \tilde{f}(P,\hat{\tau},\tau_N) = 0$ the implicit function theorem in its real analytic version applied to $\nabla_{\hat{\tau}} \tilde{f} \in C^\omega(O^* \times \hat{U} \times I, \mathbb{R}^{N-1})$ (by Satz 3 in [16]) yields:

There exist neighborhoods $\hat{U}_1 \subset \hat{U}$ of $\hat{\tau}$, $O_1^* \subset O^*$ of P^* and $I_1 \subset I$ of τ_N^*, where O_1^* and I_1 have to be chosen sufficiently small depending on the choice of $\hat{U}_1$, and some function $F \in C^\omega(O_1^* \times I_1, \hat{U}_1)$ such that there holds $F(P^*, \tau_N^*) = \hat{\tau}^*$ and

$$\begin{aligned} &\{(P, \hat{\tau}, \tau_N) \in O_1^* \times \hat{U}_1 \times I_1 \mid \nabla_{\hat{\tau}} \tilde{f}(P, \hat{\tau}, \tau_N) = 0\} \\ = &\{(P, F(P, \tau_N), \tau_N) \mid (P, \tau_N) \in O_1^* \times I_1\} = \operatorname{graph}(F). \end{aligned} \tag{11.18}$$

Now let some $P \in O_1^*$ be arbitrarily chosen, then a point $(F(P, \tau_N), \tau_N) \in \hat{U}_1 \times I_1$ is contained in $K(\tilde{f}(P, \cdot\,)) \cap (\hat{U}_1 \times I_1)$ if and only if it satisfies $\nabla_\tau \tilde{f}(P, F(P, \tau_N), \tau_N) = 0$, which is by (11.18) equivalent to

$$\Phi(P, \tau_N) := \frac{\partial \tilde{f}}{\partial \tau_N}(P, F(P, \tau_N), \tau_N) = 0. \tag{11.19}$$

Now Φ, being an element of $C^\omega(O_1^* \times I_1, \mathbb{R})$, can be developed into a power series about the point (P^*, τ_N^*):

$$\Phi(P, \tau_N) = \sum_{\alpha \in \mathbb{N}_0^{3N+9}} \sum_{j=0}^{\infty} \Phi_{\alpha,j} (P - P^*)^\alpha \, (\tau_N - \tau_N^*)^j$$

for any pair $(P, \tau_N) \in O_1^* \times I_1$. Now we prove the existence of some positive integer q with

$$\Phi_{0,0} = \Phi_{0,1} = \ldots = \Phi_{0,q-1} = 0, \quad \text{but} \quad \Phi_{0,q} \neq 0. \tag{11.20}$$

Firstly we note that there holds in particular

$$\Phi_{0,0} = \Phi(P^*, \tau_N^*) = \frac{\partial \tilde{f}}{\partial \tau_N}(P^*, F(P^*, \tau_N^*), \tau_N^*) = 0$$

due to $(F(P^*, \tau_N^*), \tau_N^*) = (\hat{\tau}^*, \tau_N^*) \in K(\tilde{f}(P^*, \cdot\,))$. But on the other hand if $\Phi_{0,j}$ would vanish for any $j \in \mathbb{N}_0$, then we could conclude that $\Phi(P^*, \cdot\,) \equiv 0$ on I_1, which would imply together with (11.18) that $\{(F(P^*, \tau_N), \tau_N) \mid \tau_N \in I_1\}$ would be an analytic curve of critical points of $\tilde{f}(P^*, \cdot\,)$ passing through the point $(F(P^*, \tau_N^*), \tau_N^*) = \tau^*$ which is contained in $K_s(f(P^*, \cdot\,))$. But this is exactly the *hypothesis* at the beginning of Chapter 10 which is shown to lead to a contradiction in the course of Chapters 10 and 11, in order to prove the finiteness of the set $K_s(f(P^*, \cdot\,)) \cong \mathcal{M}_s(\Gamma^*)$. Thus there exists in fact some integer $q \geq 1$ satisfying the assertion in (11.20). Hence, all requirements of Weierstrass' preparation theorem (see [25], p. 152) are satisfied which yields the existence of some neighborhood $O_2^* \times I_2$ of (P^*, τ_N^*), of some Weierstrass polynomial

$$W(P, \tau_N) = (\tau_N - \tau_N^*)^q + a_{q-1}(P)\,(\tau_N - \tau_N^*)^{q-1} + \ldots + a_1(P)\,(\tau_N - \tau_N^*) + a_0(P),$$

defined on $O_2^* \times I_2$, with coefficients $a_j \in C^\omega(O_2^*, \mathbb{R})$ satisfying $a_j(P^*) = 0$, for $j = 0, \ldots, q-1$, and of some non-vanishing function $V \in C^\omega(O_2^* \times I_2, \mathbb{R})$ which fulfill

$$\Phi(P, \tau_N) = \frac{\Phi_{0,q}}{V(P, \tau_N)} W(P, \tau_N) \qquad \forall\, (P, \tau_N) \in O_2^* \times I_2. \tag{11.21}$$

Thus the zeroes of $\Phi(P, \cdot\,)$ and $W(P, \cdot\,)$ coincide on I_2 and their number is bounded by the degree $q \equiv q(P^*, \tau^*)$ of $W(P, \cdot\,)$ uniformly for any fixed $P \in O_2^*$. Thus together with Corollary 2.4, (11.19) and $I_2 \subset I_1$ we infer that

$$\begin{aligned} &\sharp(K_s(f(P, \cdot\,)) \cap (\hat{U}_1 \times I_2)) \leq \sharp(K(\tilde{f}(P, \cdot\,)) \cap (\hat{U}_1 \times I_2)) \\ &= \text{number of zeroes of } \Phi(P, \cdot\,) \text{ on } I_2 \leq q(P^*, \tau^*) \qquad \forall\, P \in O_2^*. \end{aligned}$$

Now we know that $K_s(f(P^*, \cdot\,))(\neq \emptyset)$ consists of only finitely many points $\{\tau^j\}_{j=1,\ldots,J}$ that satisfy Case 1 or 2, thus that possess disjoint open neighborhoods U^j in T for which hold:

$$\sharp(K_s(f(P, \cdot\,)) \cap U^j) \leq q^j(P^*, \tau^j) \qquad \forall\, P \in O^j, \tag{11.22}$$

for $j = 1, \ldots, J = \sharp(K_s(f(P^*, \cdot\,)))$, where q^j are degrees of Weierstrass polynomials depending on P^* and τ^j only and where O^j are sufficiently small neighborhoods of P^* in $\mathbb{R}^{3N+9}$. As the disjoint union $\mathring{\bigcup}_{j=1}^J U^j =: U$ is an open neighborhood of $K_s(f(P^*, \cdot\,))$ in T the above lemma yields the existence of some neighborhood O of P^* in $\mathbb{R}^{3N+9}$ such that there holds $K_s(f(P, \cdot\,)) \subset U \;\; \forall\, P \in O$, i.e. we have:

$$K_s(f(P, \cdot\,)) = \mathring{\bigcup}_{j=1}^J K_s(f(P, \cdot\,)) \cap U^j \qquad \forall\, P \in O. \tag{11.23}$$

Now choosing this neighborhood O of P^* sufficiently small such that there holds also $O \subset \bigcap_{j=1}^J O^j$ we obtain by (11.22) and (11.23):

$$\begin{aligned} \sharp(K_s(f(P, \cdot\,))) &= \sum_{j=1}^J \sharp(K_s(f(P, \cdot\,)) \cap U^j) \\ &\leq \sum_{j=1}^J q^j(P^*, \tau^j) =: \beta(P^*) \qquad \forall\, P \in O. \end{aligned}$$

Thus together with the correspondences $K_s(f(P, \cdot\,)) \cong \mathcal{M}_s(\Gamma(P))$ via $\psi(P, \cdot\,)$ and $P \longleftrightarrow \Gamma(P)$ we finally obtain the claimed result of Theorem 1.2.

$\diamond$

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