The homology of Moduli Spaces of Riemann Surfaces

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Moduli spaces of Riemann surfaces

Introduction A motivating question would be the following: How can one classify the complex structures on a two dimensional manifold F? The first huge step towards a satisfactory answer, is the construction of the moduli space $\mathfrak{M}$. Its underlying points are in one-to-one correspondence with the set of equivalence classes of complex structures. The study of these moduli spaces relates topology, geometry, algebra and mathematical physics. We are interested in the homology of these moduli spaces and their harmonic compactifications.

The moduli spaces $\mathfrak{M}^\bullet_g(p,q)$ and $\mathfrak{M}_g(p,q)$ Fix $g \geq 0$, $p \geq 1$ and $q \geq 1$. Our data $\mathcal{D}$ for a surface F consists of

(1) a Riemann surface F of genus g with $p+q$ disjointly embedded discs removed, thus $\partial F \cong \coprod_{p+q} \mathbb{S}^1$;

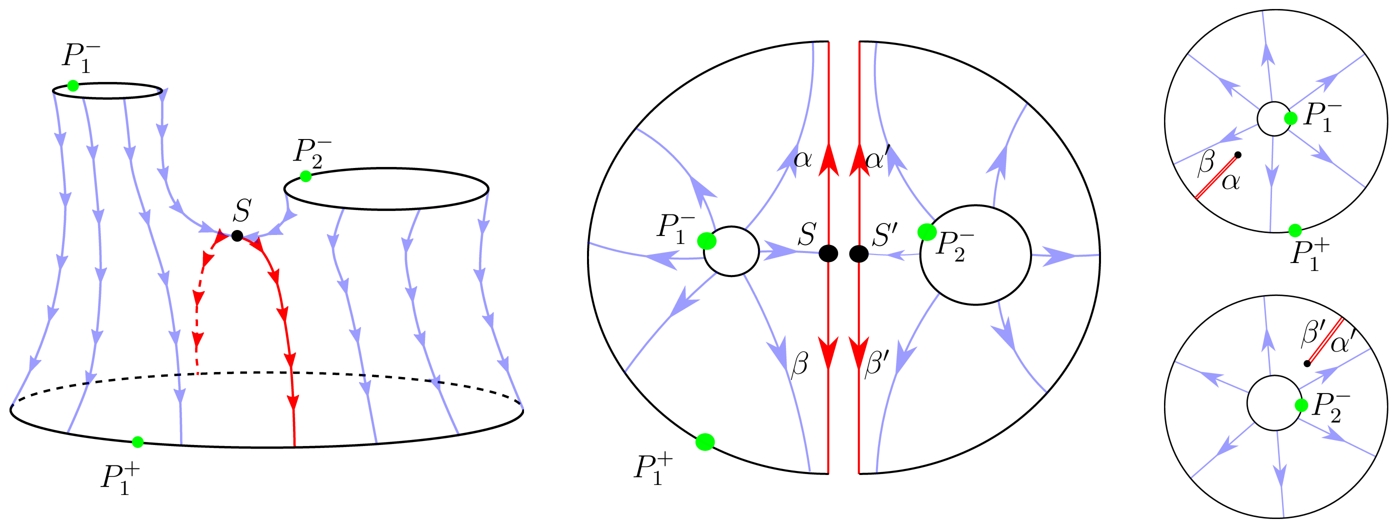
(2) p enumerated incoming boundary circles $C_1^-, \ldots, C_p^- \subset \partial F$ each of which has a marked point P_i^-;

(3) q enumerated outgoing boundary circles $C_1^+, \ldots, C_q^+ \subset \partial F$ each of which has a marked point P_i^+.

Two surfaces $[F,\mathcal{D}]$ and $[F',\mathcal{D}']$ are equivalent if and only if there is a bihomolorphic map $\varphi\colon F \longrightarrow F'$ respecting the structure. The set of equivalence classes embody the moduli space of Riemann surfaces $\mathfrak{M}^\bullet_g(p,q)$. Forgetting the marked points on the outgoing boundaries yeilds another moduli space denoted by $\mathfrak{M}_g(p,q)$. Moreover, we obtain a torus bundle $\mathbb{T}^q \longrightarrow \mathfrak{M}^\bullet_g(p,q) \longrightarrow \mathfrak{M}_g(p,q)$.

Hilbert uniformization A method providing a comfortable model for $\mathfrak{M}^\bullet_g(p,q)$ and $\mathfrak{M}_g(p,q)$ is introduced in [Böd]. In order to ease the discussion of the uniformization process, we provide a pictorial example below, where $g=0$, $p=2$ and $q=1$. Given a complex surface $[F] \in \mathfrak{M}^\bullet_g(p,q)$ there is a unique harmonic potential $u\colon F \longrightarrow \mathbb{R}_{\geq 0}$ that is constant on all boundaries with $u=0$ on the outgoing boundaries. The flow of steepest descent is drawn in light blue. It has finitely many critical points $S_1,\ldots,S_k$ in the interour of F. The union of all the flow lines leaving a critical point constitute the critical graph K drawn in red.

Observe that $F-K$ consist of exactly p components that retract onto exactly p annuli by following the flowlines backwards. The process of "straightening the remaining flow lines" defines an injective homolorphic map w from $F-K$ into p complex planes. The image are p annuli $\mathbb{A}$ minus a finite number of radial half-rays ending in the outer boundary of each annulus; this we call a radial slit configuration.

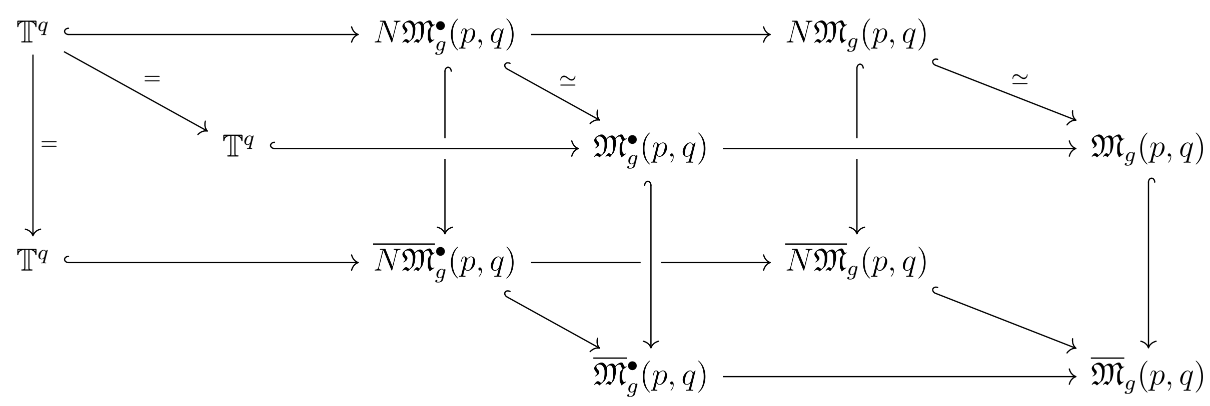


The space of such maps w is denoted by $\mathfrak{H}^\bullet_g(p,q)$. It is a bundle $\mathfrak{H}^\bullet_g(p,q) \xrightarrow{\simeq} \mathfrak{M}^\bullet_g(p,q)$ and the choices we made constitute the fibre which is contractible. The space $\mathfrak{H}^\bullet_g(p,q)$ is homeomorphic to the space of admissible radial slit configurations denoted by $\mathfrak{Rad}^\bullet_g(p,q)$. Analogous results hold for the moduli spaces with unmarked outgoing boundary $\mathfrak{M}_g(p,q)$.

The harmonic compactification

The harmonic compactification The space of radial slit configurations $\mathfrak{Rad}^\bullet_g(p,q)$ is a model for the moduli space of Riemann surfaces $\mathfrak{M}^\bullet_g(p,q)$. It is not compact; but allowing certain degenerations of handles and outgoing boundaries (such that the harmonic potential u is still defined), we obtain the harmonic compactification $\mathfrak{M}^\bullet_g(p,q) \subset \overline{\mathfrak{M}}^\bullet_g(p,q)$. In [EK], it is identified with a space of Sullivan diagrams $\overline{\mathfrak{M}}^\bullet_g(p,q) \cong \mathscr{S}\mathscr{D}^\bullet_g(p,q)$. The spaces of Sullivan diagrams are studied in string topology, e.g. they classify all natural (higher string-) operations on the Hochschild complex of symmetric Frobenius algebras, see [Wah].

Observe that the moduli space deformation retracts onto the subspace $N\mathfrak{M}^\bullet_g(p,q) \subset \mathfrak{M}^\bullet_g(p,q)$ consisting of all moduli where each boundary component has circumference one. Allowing the same degenerations of handles as before, we obtain again a harmonic compactification $N\mathfrak{M}^\bullet_g(p,q) \subset \overline{N\mathfrak{M}}^\bullet_g(p,q) \subset \overline{\mathfrak{M}}^\bullet_g(p,q)$ and similarly for $\mathfrak{M}_g(p,q)$. The relationship between the spaces discussed so far is shown in the following diagram where all rows but the row in the lower front are torus fibrations.



Sewing a fixed surface of genus one with exactly one incoming and one outgoing boundary to the first incoming boundary of a modulus $[F]$ increases the genus of $[F]$ by one. This operation defines the so called stabilization map of moduli spaces. Moreover, there are stabilization maps on the level of harmonic compactifications that respect the fibrations (being the identity on the fibre) as well as the inclusions in the diagram above. Let us present our results on the homotopy type of the harmonic compactifications:

Theorem (B.–Egas 2016). *Given parameters $g \geq 0$, $p=1$, $q \geq 1$ and q integers $c_i > 1$ with $c = c_1 + \ldots + c_q$.*

1. *The harmonic compactifications $\overline{\mathfrak{M}}^\bullet_g(p,q)$ and $\overline{\mathfrak{M}}_g(p,q)$ are highly connected with respect to the number of outgoing boundaries i.e.*

$$\pi_i(\overline{\mathfrak{M}}^\bullet_g(p,q)) = \pi_i(\overline{\mathfrak{M}}_g(p,q)) = 0 \quad \text{for } 0 \leq i \leq q-2\,.$$

2. *The stabilization maps*

$$\overline{\mathfrak{M}}^\bullet_g(p,q) \longrightarrow \overline{\mathfrak{M}}^\bullet_{g+1}(p,q) \quad \text{and} \quad \overline{\mathfrak{M}}_g(p,q) \longrightarrow \overline{\mathfrak{M}}_{g+1}(p,q)$$

are $(g+q-2)$-connected.

3. *There are classes of infinite order $\gamma_q \in H_{4q-1}(\overline{\mathfrak{M}}^\bullet_g(1,q);\mathbb{Z})$ and $\omega_{(c_1,\ldots,c_m)} \in H_{2c-1}(\overline{\mathfrak{M}}^\bullet_0(1,\sum c_i);\mathbb{Z})$. All these correspond to non-trivial higher string topology operations.*

Our next results describe the homotopy type of the so called stable moduli spaces $\overline{\mathfrak{M}}^\bullet_\infty(p,q)$ resp. $\overline{\mathfrak{M}}_\infty(p,q)$ and their harmonic compactifications. The inclusion into the stable moduli space yeilds isomorphisms in homology in a range increasing with g. A homology class is called stable, if it survives the inclusion into the stable moduli space.

Results on the stable and unstable homology

Theorem (B. 2017⁺). *Given parameters $g \geq 0$, $p=1$ and $q \geq 1$, denote the classifying maps of the torus bundle by $\vartheta_g\colon \overline{N\mathfrak{M}}_g(p,q) \longrightarrow (\mathbb{C}P^\infty)^q$.*

1. *The maps ϑ_g and the stabilization maps are $(g-2)$ connected.*
2. *In particular, $\overline{N\mathfrak{M}}^\bullet_\infty(p,q) \longrightarrow \overline{N\mathfrak{M}}_\infty(p,q)$ is homotopy equivalent to $E\mathbb{T}^q \longrightarrow B\mathbb{T}^q \simeq (\mathbb{C}P^\infty)^q$.*

In [Wah] it was shown that all stable homology classes in $\mathfrak{M}^\bullet_g(p,q)$ vanish in $\overline{\mathfrak{M}}^\bullet_g(p,q)$. Observe that this is recovered by our theorem above because the inclusion $\mathfrak{M}^\bullet_\infty(p,q) \subset \overline{\mathfrak{M}}^\bullet_\infty(p,q)$ factors through the contractible space $\overline{N\mathfrak{M}}^\bullet_\infty(p,q)$. For stable homology classes in $\mathfrak{M}_g(p,q)$ the situation is different:

Theorem (B. 2017⁺). *The inclusion into the stable harmonic compactification $\mathfrak{M}_\infty(p,q) \subset \overline{N\mathfrak{M}}_\infty(p,q)$ induces the canonical inclusion of rings*

$$H^*(\mathfrak{M}_\infty(p,q);\mathbb{Q}) \cong \mathbb{Q}[u_1,\ldots,u_q,\kappa_i \mid i \geq 1] \longleftarrow \mathbb{Q}[u_1,\ldots,u_q] \cong H^*((\mathbb{C}P^\infty)^q;\mathbb{Q})\,.$$

In particular, the inclusion $\mathfrak{M}_\infty(p,1) \longrightarrow \overline{\mathfrak{M}}_\infty(p,1) \simeq \mathbb{C}P^\infty$ induces a non-trivial map in homology.

The last statement follows from the univeral coefficient theorem and the fact that $\overline{N\mathfrak{M}}_\infty(p,1) \simeq \overline{\mathfrak{M}}_\infty(p,1)$.

The moduli spaces $\mathfrak{M}^m_{g,1}$ In the remainder of this poster, we discuss the unstable homology of moduli spaces of Riemann surfaces with one parametrized incoming and with $m \geq 0$ unenumerated unparametrized outgoing boundaries denoted by $\mathfrak{M}^m_{g,1}$. Observe that the forgetful map $\mathfrak{M}_g(1,m) \longrightarrow \mathfrak{M}^m_{g,1}$ is a $m!$-fold covering. Moreover, the disjoint union $\mathfrak{M} := \coprod_{g,m} \mathfrak{M}^m_{g,1}$ is an E_2-space.

In [BB], Bödigheimer and the author give a partial description of the unstable homology of the moduli spaces $\mathfrak{M}^m_{g,1}$ by means of generators and relations. Let us discuss three examples.

Unstable homology via small models As before, the Hilbert uniformization yeilds a space of radial slit domains $\mathfrak{Rad}^m_{g,1}$ modelling the moduli space $\mathfrak{M}^m_{g,1}$. It is a combinatorial, relative manifold, i.e., $\mathfrak{Rad}^m_{g,1} \cong \mathbb{P}-\mathbb{P}'$ with $(\mathbb{P},\mathbb{P}')$ a pair of compact cell complexes. The homology of $\mathfrak{M}^m_{g,1}$ is therefore Poincaré dual to the cohomology of $\mathbb{P}/\mathbb{P}'$. Using this description of the homology of $\mathfrak{M}^m_{g,1}$, explicit computations for $2g+m \leq 6$ were carried out by Abhau, Bödigheimer, Ehrenfried, Hermann, Mehner, Wang and the author. This way, one obtains explicit generators and relations using the various homology operations.

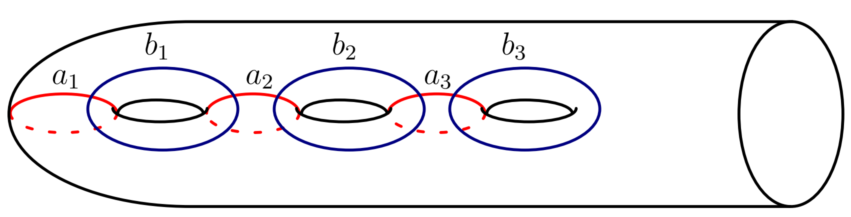
Unstable homology via braid groups I The moduli space $\mathfrak{M}^m_{0,1}$ is the space of m undistinguishable particles in an open disc. Thus, $Br_m = \pi_1(\mathfrak{M}^m_{0,1})$ is the braid group on m stands and $H_*(Br_m) \cong H_*(\mathfrak{M}^m_{0,1})$.

Introducing a new puncture near the boundary curve of a closed surface $F_{g,1}$ of genus g with one boundary component induces a map $\mathfrak{M}^m_{g,1} \longrightarrow \mathfrak{M}^{m+1}_{g,1}$ of moduli spaces. The induced map in homology $H_0(\mathfrak{M}^1_{0,1};\mathbb{Z}) \otimes H_*(\mathfrak{M}^m_{g,1};\mathbb{Z}) \longrightarrow H_*(\mathfrak{M}^{m+1}_{g,1};\mathbb{Z})$ is the operadic multiplication with the generator in $H_0(\mathfrak{M}^1_{0,1})$. It is split-injective by [BT1]. Using the braid group on two strands, we obtain infinite families of non-trivial (unstable) homology classes.

Theorem (B. 2015⁺). *The generator $b \in H_1(Br_2;\mathbb{F}_2) \cong H_1(\mathfrak{M}^2_{0,1};\mathbb{F}_2)$ spans a polynomial ring $\mathbb{F}_2[b]$ inside $H_*(\mathfrak{M};\mathbb{F}_2)$. Regarding $H_*(\mathfrak{M};\mathbb{F}_2)$ as a module over $\mathbb{F}_2[b]$, it is torsion free.*

More results on the unstable homology

Unstable classes via braid groups II It is well known that the k^{th} braid group Br_k is isomorphic to the mapping class group $\Gamma^k_{0,1}$ of a disc with k permutable punctures. Sending the braid generators σ_i to certain Dehn twists, [BT2] construct more families of maps from Br_k to the mapping class group $\Gamma^m_{g,1}$ of a surface of genus g with one boundary component and m permutable punctures. Let us review one of these. The map $\phi_g\colon Br_{2g} \longrightarrow \Gamma^0_{g,1}$ sends the generators $\sigma_1,\ldots,\sigma_{2g-1}$ to the Dehn twists along the simple closed curves $a_1,b_1,\ldots,a_g,b_g$ drawn red and blue in picture below.



The stable version $\phi_\infty\colon Br_\infty \longrightarrow \Gamma^0_{\infty,1}$ comes from a map of double-loop spaces that is null-homotopic [BT2]. Therefore, ϕ_g is the trivial map in homology in the stable range. The same is true for most maps constructed in [BT2]. However, it turns out that some of these are non-trivial in the unstable range.

Proposition (B. 2016⁺). *For $g \leq 2$, the map ϕ_g induces a split injection in homology outside the stable range. Moreover, we have a canonical map $\psi_2\colon Br_6 \longrightarrow \Gamma^0_{2,1}$, inducing a split injection $\mathbb{Z}/3\mathbb{Z} \cong H_4(Br_6;\mathbb{Z}) \longrightarrow H_4(\Gamma^0_{2,1};\mathbb{Z})$.*

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