Nef, big, ample

X = projective variety

$Z_1(X)$ = free abelian group generated by integral curves

$\rightsquigarrow \quad Z_1(X) \times \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$
$\qquad (C \quad , \quad L \quad) \longmapsto \deg(L_{/C})$

$\rightsquigarrow$ numerical equivalence on $Z_1(X)$ and $\mathrm{Pic}(X)$: "$\sim$"
E.g. $L \sim L' \iff \deg(L_{/C}) = \deg(L'_{/C}) \ \forall C$

$\rightsquigarrow N_1(X) := Z_1(X)/_\sim \; ; \; N^1(X) := \mathrm{Pic}(X)/_\sim \quad (= NS(X))$

Then $N_1(X) \times N^1(X) \to \mathbb{Z}$ is non-degenerate

$\rightsquigarrow$ vector spaces / $\mathbb{Q}, \mathbb{R}$: $N_1(X)_{\mathbb{Q},\mathbb{R}}$, $N^1(X)_{\mathbb{Q},\mathbb{R}}$

<u>Fact</u>: $N^1(X)_{\mathbb{Q}}$ (and hence $N_1(X)_{\mathbb{Q}}$) is finite-dim.

<u>Cone of curves</u>: $NE(X)_{\mathbb{R}} \subset N_1(X)_{\mathbb{R}}$

$$:= \{ \textstyle\sum a_i \cdot C_i \mid a_i \geq 0 \}$$

It is a convex cone, but in general not closed!

$\rightsquigarrow \overline{NE(X)}_{\mathbb{R}}$

$NE(X)_{\mathbb{Q}} \subset \overline{NE(X)}_{\mathbb{R}}$ is dense

If D Cartier divisor or line bundle

$\rightsquigarrow D_{>0} := \{ \alpha = \sum a_i C_i \mid (D.\alpha) > 0 \} \subset N_1(X)_{\mathbb{R}}$
"half space"

Analogously : $D_{<0}$, $D_{=0}$, $D_{\geq 0}, \dots$

<u>Kleiman's ampleness criterion</u> $D \in \mathrm{Ca}(X)$ is
ample $\iff \overline{NE(X)} \setminus \{0\} \subset D_{>0}$

(The proof is based on the Nakai-Moishezon criterion.)

<u>Remark</u>: • In particular this shows that if $L_1, L_2 \in \mathrm{Pic}(X)$ are numerically equivalent ($L_1 \sim L_2$), then
L_1 ample $\iff$ L_2 ample

• One could define the 'ample cone in $N^1(X)_{\mathbb{R}}$' as $\{ D \in N^1(X)_{\mathbb{R}} \mid \overline{NE(X)} \setminus \{0\} \subset D_{>0} \}$, but this is not often used.

<u>Def</u>: $D \in N^1(X)_{\mathbb{R}}$ is <u>nef</u> if $\overline{NE(X)} \subset D_{\geq 0}$

$\iff (C.D) \geq 0$ for all integral curves $C \subset X$.

<u>Remark</u>: • The nef cone is the closure of the ample cone.
• If L^m is globally generated for $m \gg 0$, then L is nef.

<u>Kodaira dimension</u> : X = projective, $L \in \mathrm{Pic}(X)$

$$\kappa(X,L) := \begin{cases} \max\{\dim \mathrm{Im}(\varphi_{L^m} : X \dashrightarrow \mathbb{P}^{N_m})\} & \text{if } H^0(X,L^m) \neq 0 \text{ for some } m>0 \\ -\infty & \text{else} \end{cases}$$

<u>Remark</u>: ($\nearrow$ [Ueno]):

$$\kappa(X,L) = \begin{cases} \operatorname{trdeg} Q\left(\bigoplus_{m=0}^{\infty} H^0(X,L^m)\right) - 1 & \text{― " ―} \\ -\infty \end{cases}$$

$= \kappa \iff \exists\, \alpha, \beta, m_0 \in \mathbb{N}$:

$$\alpha m^{\kappa} \leq \dim H^0(X, L^{m \cdot m_0}) \leq \beta \cdot m^{\kappa} \quad \text{for } m \gg 0$$

(see also [Lazarsfeld])

<u>Def</u>: L is <u>big</u> if $\kappa(X,L) = \dim(X)$

<u>Remark</u>: Usually one only considers line bundles that are not only big, but also nef.
Using Kodaira's lemma (see later), one can show that

$$L \text{ nef and big} \iff L \text{ nef and } (\underbrace{L \cdot \ldots \cdot L}_{n\text{-times}}) > 0$$

$(n = \dim X)$

See [Kollár, Mori] for the complete argument.

Remark: i) Suppose $\pi: X \to Y$ is a proper morphism and $L \in \mathrm{Pic}(Y)$. Then

- L nef $\Rightarrow$ $\pi^* L$ nef
- π dominant: L nef $\Leftrightarrow$ $\pi^* L$ nef
- π dominant and generically finite: L big $\Rightarrow$ $\pi^* L$ big
- π finite: L ample $\Leftrightarrow$ $\pi^* L$ ample.

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ii) $D_1, D_2 \in \mathrm{Ca}(X)_{\mathbb{Q}}$, D_1 nef, D_2 ample

$\Rightarrow$ • $D_1 + D_2$ ample

• $D_1^{n-i} \cdot D_2^{i} \geq 0$

• $D_1 \, D_2^{n-1} > 0$ if $D_1 \not\equiv 0$.

Kawamata-Viehweg vanishing

Recall first the

Kodaira vanishing theorem : X = smooth projective $/k=\bar{k}$, char $k=0$ and $L \in \mathrm{Pic}(X)$ ample. Then

$$H^p(X, L\otimes K_X)=0 \quad \text{for } p>0.$$

Remarks : • By GAGA : $H^p_{Zar} = H^p_{an}$.

- ∃ analytic proof via curvature, positivity of forms, Nakano identity.
- ∃ algebraic proof by Deligne/Illusie that uses reduction mod p
- In [Lazarsfeld] one finds a proof that uses first a cyclic cover in order to produce a section of L and the weak Lefschetz theorem to lower dimension.
  Conversely, the Kodaira vanishing can be used to prove the weak Lefschetz! Thus, Kodaira vanishing and weak Lefschetz are essentially equivalent
- By the Lefschetz principle one can reduce to other fields

The Kawamata-Viehweg (KV) vanishing can be seen as a generalization of the Kodaira vanishing in several ways

1. L only nef+big

2. log-version

3. = 1. + 2.

⋮

singular versions

1. <u>Theorem</u>: Let L be a nef & big line bundle on a smooth projective variety $/\mathbb{C}$. Then
$$H^p(X, L\otimes K_X)=0 \quad \text{for } p>0.$$

<u>Proof</u>:

<u>1st step</u>: <u>Kodaira's lemma</u>: If L is a big line bundle on a variety X and D_0 an effective Cartier divisor on X. Then $H^0(X, L^m(-D_0))\neq 0$ for $m\gg 0$

<u>Proof</u>: By definition L big means $h^0(X,L^m)\sim m^n$, where $n=\dim X$. Then use short exact sequence
$$0\to L^m(-D_0)\to L^m\to L^m/_{D_0}\to 0$$
, which yields exact sequence
$$0\to H^0(X,L^m(-D_0))\to H^0(X,L^m)\to H^0(D_0, L^m/_{D_0})\to$$
Since $h^0(D_0, L^m/_{D_0})$ grows at most like m^{n-1}, one has $H^0(X,L^m(-D_0))\neq 0$ for $m\gg 0$. □

Note that the assertion here more precisely means: $\exists m_0, m_1$ st.
$$H^0(X, L^m(-D_0))\neq 0 \quad \text{for all } m=km_0 \text{ and } k\geq m_1.$$

<u>Corollary</u>: If L is big, then for some $m\gg 0$:
$$L^m\simeq \mathcal{O}(H+D) \quad \text{with } H \text{ ample, } D \text{ effective.}$$

<u>Proof</u>: Choose effective ample divisor H. Then Kodaira's lemma shows $H^0(X,L^m(-H))\neq 0$, i.e. $L^m(-H)\simeq\mathcal{O}(D)$ with D effective. □

From now on "L nef & big" is replaced by
"L nef & $L^m\simeq\mathcal{O}(H+D)$ with H ample, D effective"

<u>2nd step</u>: Suppose X is smooth, projective, H is ample, and $E = \sum^{k} E_i$ is a <u>snc</u> divisor (reduced!)

<u>Claim</u>. $H^p(X, \mathcal{O}(-H-E)) = 0$ for $p < \dim(X)$

<u>Proof</u>: By induction on number of components of E and dimension. For $k=0$, the assertion is Kodaira vanishing.

Write short exact sequence

$$0 \to \mathcal{O}(-E_k) \to \mathcal{O} \to \mathcal{O}_{E_k} \to 0 \qquad \text{and twist } \otimes \mathcal{O}(-H-\sum^{k-1} E_i)$$

This yields short exact sequence

$$0 \to \mathcal{O}(-H-E) \to \mathcal{O}(-H-\sum^{k-1} E_i) \to \mathcal{O}_{E_k}(-H_{|E_k} - (\sum^{k-1} E_i)_{|E_k}) \to 0$$

and hence

$$\underbrace{H^{p-1}(E_k, \mathcal{O}(-H_{|E_k} - (\sum^{k-1} E_i)_{|E_k}))}_{\substack{=0,\text{ since } H_{|E_k} \text{ ample} \\ \text{and } \sum^{k-1} E_i{}_{|E_k} \text{ again snc}}} \to H^p(X, \mathcal{O}(-H-E)) \to \underbrace{H^p(X, \mathcal{O}(-H-\sum^{k-1} E_i))}_{\substack{=0 \text{ by induction} \\ \text{hypothesis.}}}$$

□

<u>3rd step</u>: Let X be smooth, projective and $L^m = \mathcal{O}(H+E)$ with H ample and $E = \sum a_i E_i$ snc, $a_i > 0$.

<u>Claim</u>: Then $H^p(X, L \otimes K_X) = 0$ for $p > 0$

($\Leftrightarrow$ $H^p(X, L^*) = 0$ for $p < \dim X$, by Serre duality.)

<u>Proof</u>: Introduce $m_i := m \cdot \prod_{j \neq i} a_j$

Then consider the Kawamata covering

$\pi: X' \to X$ s.t. X' smooth, $\pi^* E_i = m_i E_i'$

and $\sum E_i'$ snc

Set $L' := \pi^* L$, $H' := \pi^* H$ (again ample) and $E' := \sum E_i'$

Then $mL' = H' + maE'$, where $a = \prod a_i$

Check: $ma(L' - E') = H' + m(a-1)L'$ and hence

$L' = L' - E' + E' = \frac{1}{ma}(H' + m(a-1)L') + E'$ in $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Note that $H' + m(a-1)L'$ is the sum of an ample and a nef divisor. Hence itself ample.

Thus step 2 applies: $H^p(X', L'^*) = 0$ for $p < \dim X' = \dim X$

By projection formula this yields $H^p(X, L^* \otimes \pi_* \mathcal{O}_{X'}) = 0$, $p < \dim X$.

Eventually use that $\mathcal{O}_X \to \pi_* \mathcal{O}_{X'}$ naturally splits.

(Indeed, Kawamata coverings are compositions of "roots of sections" (with $\pi_* \mathcal{O}_{X'} = \bigoplus_{i=0}^{m-1} L^{-i}$) and "roots of line bundles" (Bloch, Gieseker), which are induced by $\varphi: \mathbb{P}^N \to \mathbb{P}^N$ $[z_0 : \ldots : z_N] \mapsto [z_0^m : \ldots : z_N^m]$ and $\mathcal{O}_{\mathbb{P}^N} \to \varphi_* \mathcal{O}_{\mathbb{P}^N}$ splits.)

<u>4th step</u>: One reduces to the situation considered in 3. via resolution of singularities.
First write $L^m = \mathcal{O}(H + D)$ with D effective.
Then there exists a "resolution of D": $\pi: Y \to X$ with $\pi^* D$ a SNC divisor. (Note that this time $\pi^* D$ contains exceptional divisor.)

Now one would like to argue as follows:
If $L' := \pi^* L$, then $L'^m = \mathcal{O}(\pi^* H + \pi^* D)$.

By step 3 : $H^p(Y, L' \otimes K_Y) = 0 \quad p > 0$

$= H^p(X, L \otimes \pi_* K_Y)$ projection formula

$= H^p(X, L \otimes K_X)$ X smooth.

There are two problems here:

i) $\pi^* H$ not ample anymore

ii) For the projection formula we also need

$$R^i \pi_* K_Y = 0 \quad \text{for } i > 0 .$$

Both are taken care of by the lemma below, which in particular ensures that

$$N \pi^* H - \Sigma b_i E_i \quad \text{with } b_i > 0, \quad E_i \text{ except.}$$

and $N \gg 0$ is ample.

Then $Nm L' = \underbrace{(N \pi^* H - \Sigma b_i E_i)}_{\text{ample}} + \underbrace{(N\pi^* D + \Sigma b_i E_i)}_{\text{snc}}$. □

<u>Lemma</u>: If $\pi : Y \to X$ is a resolution of D constructed by a sequence of blow-ups with smooth center then

i) $\exists N \gg 0, b_i > 0 : N H - \Sigma b_i E_i$ ample

ii) $R^i \pi_* K_Y = 0$ for $i > 0$

Both statements are trivial for smooth blow-ups. □