Noncommutative Spaces

Koen van den Dungen

V5B8 – Selected Topics in Analysis
Summer semester 2021

This lecture course is a first introduction to the field of noncommutative geometry (NCG), largely focusing on examples of noncommutative spaces. The lectures are scheduled to take place on **Wednesdays, 14:00–16:00**.
There are no strict prerequisites, and the lectures should be accessible to all first-year Master students. However, some previous knowledge about operators on Hilbert spaces and a bit of functional analysis would be useful.

Introduction

For a classical 'space' there is a duality between the space itself and the (commutative) algebra of functions on the space. To give a precise statement: the category of compact Hausdorff topological spaces is dual to the category of unital commutative C^*-algebras. This duality suggests that the study of *non*commutative C^*-algebras should be thought of as 'noncommutative topology'. The goal of noncommutative geometry is to extend this idea further, describing not only the topology of a space but also its (differential) geometry in terms of corresponding algebraic objects, taking the following steps:

1. translate all topological and geometrical data into algebraic data;
2. show how to reconstruct the topological and geometrical data from the algebraic data;
3. generalise by allowing noncommutative 'coordinate algebras'.

This approach also allows to deal with for instance 'bad quotients', e.g. a quotient of a compact Hausdorff space by a discrete group action, for which the quotient topology may fail to separate orbits. In this case, rather than considering continuous functions on the quotient space, one may obtain more information by studying suitable *non*commutative algebras. In this way, the study of noncommutative geometry can also be fruitful for understanding 'badly behaved' classical spaces (for a brief introduction to these ideas, see e.g. [CM08]).
In these lectures, we will largely focus on the study of groupoid C^*-algebras, which can be used to describe non-Hausdorff quotient spaces, group actions on spaces, foliated spaces, and much more. We will in particular look at many examples, and aim to study their 'topological invariants' (in terms of K-theory) and/or their 'noncommutative geometry' (in terms of spectral triples).

List of topics

The lecture course will cover (a selection of) the following topics:

Noncommutative topology: Introduction to C*-algebras. Gelfand-Naimark duality between compact Hausdorff topological spaces and unital commutative C^*-algebras. References: [Mur90, Ch. 1-3], [Ped89, Ch. 4], [Dav96, Ch. I], [GVF01, Ch. 1].

Groupoid C^*-algebras: Introduction to groupoids. General construction of a (typically noncommutative) C^*-algebra from a groupoid, which can be used to describe non-Hausdorff quotient spaces, group actions on spaces, foliated spaces, and much more. Main example: the noncommutative torus, also known as the irrational rotation algebra, which is closely related to the linear foliation of the torus by lines with an irrational angle. References: [Ren80] (for a brief introduction, see also [DL10, §1-2], and for a book with emphasis on foliations, see [CC03, Ch. 1]).

K-theory: Introduction to (C^*-algebraic) K-theory. Possibly: Serre-Swan duality between vector bundles and finitely generated projective modules, and comparison with topological K-theory. Examples of K-theory of some groupoid C^*-algebras. References: [Mur90, Ch. 7], [Bla98].

Spectral triples: these are the main objects thought of as 'noncommutative spaces' equipped with additional geometric structure. Motivating example: the spectral triple of a Riemannian spin manifold (Dirac operators). Noncommutative example: transverse Dirac operator on Riemannian foliations. References: [GVF01, Ch. 9-11], [HR00], [Kor08].

Recommended literature

[Bla98] B. Blackadar, *K-theory for operator algebras*, 2nd ed., Math. Sci. Res. Inst. Publ., Cambridge University Press, 1998.

[CC03] A. Candel and L. Conlon, *Foliations II*, Graduate Studies in Mathematics, vol. 60, American Mathematical Society, 2003.

[CM08] A. Connes and M. Marcolli, *A walk in the noncommutative garden*, An Invitation to Noncommutative Geometry (M. Khalkhali and M. Marcolli, eds.), World Scientific, 2008, pp. 1–128.

[Dav96] K. Davidson, *C^*-algebras by example*, Fields Institute for Research in Mathematical Sciences Toronto: Fields Institute monographs, American Mathematical Soc., 1996.

[DL10] C. Debord and J.-M. Lescure, *Index theory and Groupoids*, Geometric and Topological Methods for Quantum Field Theory (H. Ocampo, E. Pariguan, and S. Paycha, eds.), Cambridge University Press, 2010, Online preprint arXiv:0801.3617.

[GVF01] J. Gracia-Bondía, J. Várilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, 2001.

[HR00] N. Higson and J. Roe, *Analytic K-Homology*, Oxford University Press, New York, 2000.

[Kor08] Y. A. Kordyukov, *Noncommutative geometry of foliations*, J. K-Theory **2** (2008), no. 2, 219–327.

[Mur90] G. Murphy, *C^*-algebras and Operator Theory*, Academic Press, 1990.

[Ped89] G. Pedersen, *Analysis now*, Graduate texts in mathematics, vol. 118, Springer-Verlag, 1989.

[Ren80] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer-Verlag Berlin Heidelberg, 1980.