Exercise Sheet 10

Discussed on 30.06.2021

Problem 1. Let $S_0 \hookrightarrow S$ be a thickening of locally noetherian schemes (i.e. a surjective closed immersion). Let X and Y be abelian schemes over S and let $f_0 \colon S_0 \cdot_S X \to S_0 \cdot_S Y$ be a morphism of abelian schemes over S_0. Show that there is at most one "deformation" of f_0, i.e. a homomorphism $f \colon X \to Y$ of abelian schemes over S which restricts to f_0 over S_0.

Problem 2. Let k be an alegbraically closed field of characteristic $p > 0$. In the following you may use without proof that every finite group scheme over k of order p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, μ_p or α_p.

(a) Compute $\mathrm{End}(\mu_p)$ and $\mathrm{End}(\alpha_p)$. Prove that for a supersingular elliptic curve E over k, the kernel of Frobenius $F \colon E \to E^{(p)}$ is isomorphic to α_p.

Hint: $\mathcal{O} = \mathrm{End}(E)$ is a maximal order in a quaternion algebra and has a valuation $v \colon \mathcal{O} \to \mathbb{Z} \cup \{\infty\}$ with $v(p) = 2$. Argue that the "residue field" $\mathcal{O}/\mathcal{O}_{v>0}$ cannot map to $\mathrm{End}(\mu_p)$.

(b) Show that $\mathrm{Hom}(\mu_p, \alpha_p) = 0$ and deduce that every subgroup $G \subset (\alpha_p)^m$ of order p is isomorphic to α_p.

(c) Let $X = E^m$ be the m-fold self-product of a supersingular elliptic curve E over k. Show that there is a canonical bijection

$$\{\text{closed subgroups } K \subset X \text{ of order } p\} \xrightarrow{\sim} \mathbb{P}^{m-1}(k).$$

(d) Show that for only finitely many of these K, the quotient X/K is isomorphic to $E_1 \times \cdots \times E_m$ for some elliptic curves $E_1, \dots, E_m$.

Hint: Show first that there are only finitely many possible choices of $E_1, \dots, E_m$. Then note that if a map $\phi \colon X \to E_1 \times \cdots \times E_m$ has a kernel of order p, then this kernel only depends on ϕ modulo p.

(*e) Prove a more precise version of (c): The functor parametrizing subgroups of order p is defined as

$$Q \colon S \mapsto \{\text{finite locally free closed } S\text{-subgroup schemes } K \subset X_S \text{ of degree } p \text{ over } S\},$$

where S ranges through k-schemes. Show that Q is represented by $\mathbb{P}^{m-1}$.

Hint: Show first that $\mathrm{End}(\alpha_{p,S}) = \mathcal{O}_S(S)$ for all k-schemes S. Then comes the tricky part: Say $S = \operatorname{Spec} R$ and $K = \operatorname{Spec} A$. Write $A = R \oplus I$, where I is the ideal defining the unit section $e \colon S \to K$. Consider the "linear" elements

$$L := \{x \in I \mid m_K^*(x) = x \otimes 1 + 1 \otimes x\}.$$

Using the existence of an embedding $K \hookrightarrow (\alpha_{p,S})^m$, prove that L is projective of rank 1 over R. If $L = Rx$, e.g. after localizing on $\operatorname{Spec} R$, show that $A = \bigoplus_{i=0}^{p-1} Rx^i$. Deduce that K is locally on S isomorphic to $\alpha_{p,S}$.