Variational and jump estimates for martingale paraproducts

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Lépingle's inequality

Brownian motion: a.s. $C^{1/r}$ for any $r > 2$.
Parametrization-invariant version of $C^{1/r}$ norm: *r-variation*

$$V^r_t f_t := \sup_{t(0)<\cdots<t(J)} \Big(\sum_j |f_{t(j+1)} - f_{t(j)}|^r \Big)^{1/r}.$$

Theorem (Lépingle, 1976)

Let $f = (f_t)$ *be a martingale. For* $1 < p < \infty$ *and* $2 < r$ *we have*

$$\|V^r_t f_t\|_p \leq C_{p,r} \|f\|_p.$$

- refines martingale maximal inequality: $Mf \leq f_0 + V^r_t f_t$
- quantifies martingale convergence: $V^r f_t$ finite $\implies f_t$ converges
- false for $r = 2$

Lépingle's inequality, endpoint version

Theorem (Pisier, Xu 1988/Bourgain 1989)

For $1 < p < \infty$ *we have the* jump inequality

$$J_2^p(f_t) := \sup_{\lambda>0} \|\lambda N_\lambda^{1/2} f_t\|_{L^p} \leq C_p \|f\|_{L^p},$$

where N_λ *is the* λ-jump counting function

$$N_\lambda f_t := \sup_{t(0)<\cdots<t(J)} \#\{j \mid |f_{t(j+1)} - f_{t(j)}| > \lambda\}.$$

Observation (Bourgain 1989)

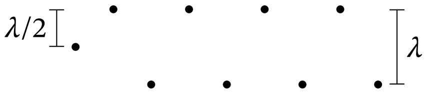
$$\|V^r f_t\|_{L^{p,\infty}} \leq C_{p,r} \sup_{\lambda>0} \|\lambda N_\lambda^{1/2} f_t\|_{L^{p,\infty}}, \quad 2 < r.$$

This + real interpolation shows that jump inequalities imply r-variational estimates in open ranges of p.

Lépingle's inequality, endpoint version, proof

λ-jump counting function is morally extremized by greedy selection of $\lambda/2$-jumps:

$$t(0) := 0, \quad t(j+1) := min\{s > t(j) \mid |f_s - f_{t(j)}| > \lambda/2\}.$$



$$\lambda N_\lambda^{1/2} \leq \lambda \Big(\sum_j \frac{|f_{t(j+1)} - f_{t(j)}|^2}{(\lambda/2)^2} \Big)^{1/2} \leq 2 \Big(\sum_j |f_{t(j+1)} - f_{t(j)}|^2 \Big)^{1/2}$$

– square function of the stopped martingale $f_{t(j)}$, bounded on L^p.

What are correct endpoint variational inequalities?

Theorem (S.J. Taylor 1972)

If (B_t) is the standard Brownian motion, then

$$\sup_{t_0<\cdots<t_J<T} \|B_{t_{j+1}} - B_{t_j}\|_{\psi(L)_j},$$

is a.s. finite with the Young function

$$\psi(t) = t^2 / \log_* \log_* t.$$

Same is true for all martingales with continuous paths, since they are reparametrizations of Brownian motion.

Question

What is the best ψ-variational estimate for general martingales?

Variational inequalities:

$$\psi(t) = t^r, r > 2.$$

Jump inequalities:

$$\psi(t) = t^2/(\log_* t)^{1+\epsilon}.$$

Lift to a rough path

Want to lift a martingale (f_t, g_t) to a rough path with Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = f_{s,u} \otimes g_{u,t}, \quad s < u < t.$$

where $s < u < t$ and $f_{s,u} = f_u - f_s$. Natural choice:

$$\begin{aligned}\mathbb{X}_{s,t} &= \int_s^t (f_{v-} - f_s)\, dg_v \\ &= \int_0^t f_{v-}\, dg_v - \int_0^s f_{v-}\, dg_v - f_s(g_t - g_s)\end{aligned}$$

Theorem (Chevyrev, Friz 2017)

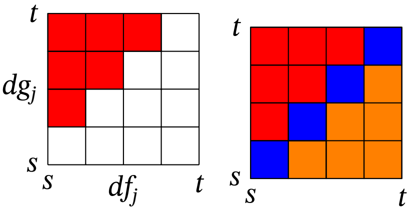
For $1 < p < \infty$ *and* $2 < r$ *we have*

$$\left\| \sup_{t_0 < \cdots < t_J} \Big(\sum_j |\mathbb{X}_{t(j),t(j+1)}|^{r/2} \Big)^{2/r} \right\|_{L^p} \le C_{p,r} \|(f,g)\|_{L^p}$$

Martingale paraproduct

For martingales $(f_j)_j$, $(g_j)_j$ and martingale differences $df_j = (f_j - f_{j-1})$ the *truncated paraproduct* (or *area process*) is defined by

$$\Pi_s^t(f,g) := \sum_{s \le j < k \le t} df_j dg_k.$$



$$(f_t - f_s)(g_t - g_s) = \Pi_s^t(f,g) + df_{s+1}dg_{s+1} + \cdots + df_t dg_t + \Pi_s^t(g,f)$$

Variational estimate for martingale paraproduct

Theorem (Do+Muscalu+Thiele 2012 (doubling), Kovač+ZK 2018 (non-doubling))

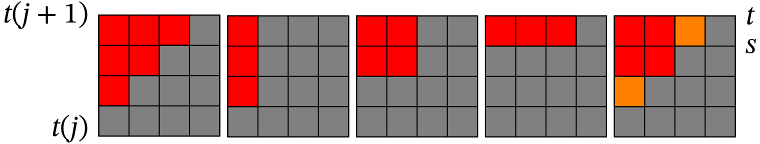
For $1 < p_1, p_2 < \infty$ *with* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ *and* $2 < r$ *we have*

$$\Big\| \sup_{t_0 < \cdots < t_J} \Big(\sum_j |\Pi_{t(j)}^{t(j+1)}(f,g)|^{r/2} \Big)^{2/r} \Big\|_{L^{p_3'}} \leq C_{p_1,p_2} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}$$

Proof idea: for $\lambda > 0$ estimate the jump counting function

$$N_\lambda := \sup_{t(0) < \cdots < t(J)} \#\{j \mid |\Pi_{t(j)}^{t(j+1)}(f,g)| > \lambda\}.$$

Stopping time with $\lambda/3$



$$\lambda N_\lambda \leq 3 \sum_{k=1} \Big| \sum_{T(k-1)<i<j\leq T(k)} df_i \, dg_j \Big| + 3 \sum_{k=1} \max_{n' \in (T(k-1),T(k)]} |f_{n'} - f_{T(k-1)}| |g_{T(k)} - g_{n'}|$$

- First term: vector-valued estimate for the paraproduct.
- Second term: vector-valued estimate for the maximal function.

Off-diagonal estimate for stochastic integrals

Corollary

Let (f_t), (g_t) be càdlàg continuous time martingales. Then for $1 < p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $2 < r$ we have

$$\Big\| \sup_{t_0 < \cdots < t_J} \Big(\sum_j |\mathbb{X}_{t(j),t(j+1)}|^{r/2} \Big)^{2/r} \Big\|_{L^{p_3'}} \leq C_{p_1,p_2,r} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

- Chevyrev+Friz 2017: diagonal case $p_1 = p_2$.
- Friz+Victoir 2006: martingales with continuous paths.