WS 2024/25

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Exercises for **Topology I**
Sheet 1

You can obtain up to 10 points per exercise (plus bonus points, where applicable).

Exercise 1. Let X be a topological space, J a set, and let $(f_j\colon \partial D^n \to X)_{j\in J}$ be a family of continuous maps. From this data, we can define the cell attachment

$$X' \coloneqq X \amalg_{f,\, J\times\partial\Delta^n} (J \times D^n)$$

with quotient map

$$p\colon X \amalg (J \times D^n) \to X'.$$

Assume we are given an open subset $V_j \subset \{j\} \times D^n$ containing the boundary $\{j\} \times \partial D^n$ for every $j \in J$. Show that $p(X \cup \bigcup_{j\in J} V_j)$ is an open subset of X'.

Remark. Recall that quotient maps of topological spaces are in general *not* open.

Exercise 2. Let $n \in \mathbb{N}$ and recall that the *n-dimensional real projective space* $\mathbb{R}\mathrm{P}^n$ is defined as the quotient of $\mathbb{R}^{1+n} \setminus \{0\}$ by the equivalence relation generated by $x \sim \lambda x$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. We saw in the previous lecture course that $\mathbb{R}\mathrm{P}^n$ can be equivalently described as the quotient of S^n by the equivalence relation generated by $x \sim -x$. Let $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ denote the quotient map.

1. Show that the cell attachment $D^{n+1} \amalg_{p,S^n} \mathbb{R}\mathrm{P}^n$ is homeomorphic to $\mathbb{R}\mathrm{P}^{n+1}$.
2. Conclude that for every $m \geq 0$ the filtration

   $$\emptyset = \mathbb{R}\mathrm{P}^{-1} \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \cdots \subset \mathbb{R}\mathrm{P}^m$$

   defines an (absolute) CW-complex structure on $\mathbb{R}\mathrm{P}^m$. Here we view $\mathbb{R}\mathrm{P}^n$ as a subspace of $\mathbb{R}\mathrm{P}^{n+1}$ via the map $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0]$.

Exercise 3. Let $n \in \mathbb{N}$ and recall that the *n-dimensional complex projective space* $\mathbb{C}\mathrm{P}^n$ is defined as the quotient of $\mathbb{C}^{1+n} \setminus \{0\}$ by the equivalence relation generated by $x \sim \lambda x$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

Give $\mathbb{C}\mathrm{P}^n$ the structure of an absolute CW-complex with one cell in every even dimension $\leq 2n$, and no other cells.

Exercise 4. Consider the subspace

$$X = \left\{\frac{1}{n} : n \geq 1\right\} \cup \{0\} \subset \mathbb{R}.$$

1. Show that the quotient Y obtained from $X \times [0,1]$ by collapsing $\big(\{0\} \times [0,1]\big) \cup \big(X \times \{0,1\}\big)$ to a single point is homeomorphic to the Hawaiian earrings (as defined in the lecture).

$*$2. (*10 bonus points*) Show that the suspension ΣX of X (i.e. the space obtained from $X \times [0,1]$ by identifying $X \times \{0\}$ to one point, and $X \times \{1\}$ to another one) is *not* homotopy equivalent to Y.

Remark. The space Y in part 1 is called the *reduced suspension* $\tilde{\Sigma} X$ of X. One can show that the natural collapse map from the ordinary suspension to the reduced one is a homotopy equivalence for every CW-complex (with an arbitrary chosen basepoint); the bonus exercise in particular shows that this is no longer true for general based topological spaces.

Please hand in your solutions by **October 21, 10am** via eCampus.