Stable homotopical algebra and Γ-spaces

By STEFAN SCHWEDE

Fakultät für Mathematik, Universität Bielefeld, 33615 *Bielefeld, Germany*,
e-mail: schwede@mathematik.uni-bielefeld.de

Introduction

In this paper we advertise the category of Γ-spaces as a convenient framework for doing 'algebra' over 'rings' in stable homotopy theory. Γ-spaces were introduced by Segal [**Se**] who showed that they give rise to a homotopy category equivalent to the usual homotopy category of connective (i.e. (-1)-connected) spectra. Bousfield and Friedlander [**BF**] later provided model category structures for Γ-spaces. The study of 'rings, modules and algebras' based on Γ-spaces became possible when Lydakis [**Ly**] introduced a symmetric monoidal smash product with good homotopical properties. Here we develop model category structures for modules and algebras, set up (derived) smash products and associated spectral sequences and compare simplicial modules and algebras to their Eilenberg–MacLane spectra counterparts.

There are other settings for ring spectra, most notably the S-modules and S-algebras of [**EKMM**] and the symmetric spectra of [**HSS**], each of these with its own advantages and disadvantages. We believe that one advantage of the Γ-space approach is its simplicity. The definitions of the stable equivalences, the smash product and the 'rings' (which we call Gamma-rings) are given on a few pages. Another feature is that Γ-spaces nicely reflect the idea that spectra are a homotopical generalization of abelian groups, that the smash product generalizes the tensor product and that algebras over the sphere spectrum generalize classical rings. There is an Eilenberg–MacLane functor H which embeds the category of simplicial abelian groups as a full subcategory of the category of Γ-spaces. The embedding has a left adjoint, left inverse which on cofibrant objects models spectrum homology. Similarly, simplicial rings embed fully faithfully into Gamma-rings. We give a quick proof (see Section 4) of the fact that modules over a simplicial ring B have the same homotopy theory as modules over the associated Eilenberg–MacLane Gamma-ring HB, and similarly for algebras when B is commutative. In Appendix B we introduce a stable model category structure for topological Γ-spaces, i.e. where simplicial sets are replaced by actual topological spaces.

One intrinsic limitation of our approach comes from the fact that Γ-spaces can only represent connective spectra. This rules out applications in certain areas of stable homotopy theory, but it is no essential restriction for the purpose of algebraic K-theory, topological Hochschild homology and topological cyclic homology. Also, even though we consider commutative Gamma-rings, and many E_∞-ring spectra can be represented this way, it seems unlikely that the homotopy category of commutative Gamma-rings is equivalent to that of connective E_∞-ring spectra.

This paper is a part of the author's thesis [**Sch2**], with some modifications. The most important one is the shift from the Bousfield–Friedlander to the Quillen model category structure for Γ-spaces (see Theorem 1·5). The weak equivalences are the same in both cases, they are the maps inducing isomorphisms on stable homotopy groups. However, the fibrations and cofibrations differ and working with the Quillen model category structure turned out to be more convenient. For example, the proof of the equivalence of homotopy theories of $H\mathbb{Z}$-algebras and simplicial rings (Theorem 4·5) is quite simple. We emphasize that Lydakis [**Ly**] works with the more general cofibrations in the sense of Bousfield and Friedlander. So some results of [**Ly**] on the compatibility of the smash product with stable equivalences are stronger than the versions we are using here.

1. *Review of Γ-spaces*

The category of Γ-spaces was introduced by Segal [**Se**], who showed that it has a homotopy category equivalent to the usual homotopy category of connective spectra. Bousfield and Friedlander [**BF**] considered a larger category of Γ-spaces in which the ones introduced by Segal appeared as the special Γ-spaces. Their category admits a closed simplicial model category structure with a notion of stable weak equivalences giving rise again to the homotopy category of connective spectra. Then Lydakis [**Ly**] showed that Γ-spaces admit internal function objects and a symmetric monoidal smash product with good homotopical properties.

Γ^{op} is a skeletal category of the category of finite pointed sets. There is one object $n^+ = \{0, 1, \ldots, n\}$ for every non-negative integer n and morphisms are the maps of sets which send 0 to 0. The category Γ^{op} is equivalent to the opposite of Segal's category Γ [**Se**]. A Γ-space is a covariant functor from Γ^{op} to the category of simplicial sets taking 0^+ to a one point simplicial set. A morphism of Γ-spaces is a natural transformation of functors. We denote the category of Γ-spaces by $\mathscr{GS}$. We sometimes need to talk about Γ-sets, by which we mean pointed functors from Γ^{op} to the category of pointed sets. Every Γ-space can be viewed as a simplicial object of Γ-sets. We choose a symmetric monoidal smash product functor $\wedge \colon \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ and extend it to one for all pointed sets. We denote by $\mathbb{S}$ the inclusion of Γ^{op} into the category of simplicial sets. The spectrum associated to the Γ-space $\mathbb{S}$ is the sphere spectrum (see below). The representable Γ-spaces $\Gamma^n = \Gamma^{\mathrm{op}}(n^+, -)$ play a role analogous to that of the standard simplices in the category of simplicial sets. Γ^1 is isomorphic to $\mathbb{S}$. If X is a Γ-space and K a pointed simplicial set, a new Γ-space $X \wedge K$ is defined by $(X \wedge K)(n^+) = X(n^+) \wedge K$.

There are three kinds of hom object for Γ-spaces X and Y. There is the set of morphisms (natural transformations) $\mathscr{GS}(X, Y)$. Then there is a simplicial hom set $\mathrm{hom}\,(X, Y)$, defined by

$$\mathrm{hom}\,(X, Y)_i = \mathscr{GS}(X \wedge (\Delta^i)^+, Y),$$

where the '+' denotes a disjoint basepoint. In this way $\mathscr{GS}$ becomes a simplicial category. Finally, Lydakis [**Ly**, definition 2·1] defines an internal hom Γ-space $\mathrm{Hom}\,(X, Y)$ by

$$\mathrm{Hom}\,(X, Y)\,(n^+) = \mathrm{hom}\,(X, Y_{n^+\wedge}),$$

where $Y_{n^+\wedge}(m^+) = Y(n^+\wedge m^+)$.

A Γ-space X can be prolonged, by direct limit, to a functor from the category of pointed sets to pointed simplicial sets. By degreewise evaluation and formation of the diagonal of the resulting bisimplicial sets, it can furthermore be promoted to a functor from the category of pointed simplicial sets to itself [**BF**, section 4]. This prolongation process has another description as the following coend [**MacL**, IX·6]. If X is a Γ-space and K a pointed simplicial set, the value of the extended functor on K is given by

$$\int^{n^+ \in \Gamma^{\mathrm{op}}} K^n \wedge X(n^+).$$

The extended functor preserves weak equivalences of simplicial sets [**BF**, proposition 4·9] and is automatically simplicial, i.e. it comes with coherent natural maps $K \wedge X(L) \to X(K \wedge L)$. We will not distinguish it notationally from the original Γ-space.

A spectrum X in the sense of [**BF**, definition 2·1] consists of a sequence of pointed simplicial sets X_n for $n = 0, 1, \ldots$, together with maps $S^1 \wedge X_n \to X_{n+1}$. A map of spectra $X \to Y$ consists of maps $X_n \to Y_n$ strictly commuting with the suspension maps. Since a Γ-space extends to a simplicial functor from all pointed simplicial sets, it defines a spectrum whose n-th term is the Γ-space evaluated at $S^n = S^1 \wedge \cdots \wedge S^1$ (n factors). Consider for example the Γ-space $\mathbb{S} \wedge K$ for a pointed simplicial set K. This Γ-space prolongs to the functor which smashes a pointed simplicial set with K. So the associated spectrum has $S^n \wedge K$ as its nth term, i.e. $\mathbb{S} \wedge K$ represents the suspension spectrum of K. Theorem 5·8 of [**BF**] states that the spectrum construction induces an equivalence of the homotopy category of Γ-spaces with the homotopy category of connective spectra. Although our paper does not formally depend on the comparison of the homotopy theories of Γ-spaces and spectra, this result is an important motivation for the study of Γ-spaces.

The homotopy groups of a Γ-space X are those of the associated spectrum,

$$\pi_n X = \mathrm{colim}_i\, \pi_{n+i} |X(S^i)|.$$

These groups are always trivial in negative dimensions. A map of Γ-spaces is called a *stable equivalence* if it induces isomorphisms on homotopy groups. An example of a stable equivalence of Γ-spaces is the map $\Gamma^1 \wedge n^+ \to \Gamma^n$ induced by the n projections $n^+ \to 1^+$. After prolongation to a pointed simplicial set K, this map includes the wedge of n copies of K into the product, which is an equivalence in the stable range.

In [**Ly**, theorem 2·2], Lydakis introduces a smash product $X \wedge Y$ for Γ-spaces X and Y. First, an external smash product $X \,\tilde{\wedge}\, Y$ is defined as the bi-Γ-space

$$(X \,\tilde{\wedge}\, Y)(k^+, l^+) = X(k^+) \wedge Y(l^+).$$

The internal smash product is then obtained by left Kan extension along the smash product functor $\wedge \colon \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$,

$$(X \wedge Y)(n^+) = \mathrm{colim}_{k^+ \wedge l^+ \to n^+} X(k^+) \wedge Y(l^+).$$

The smash product is thus characterized by the universal property that Γ-space maps $X \wedge Y \to Z$ are in bijective correspondence with maps

$$X(k^+) \wedge Y(l^+) \to Z(k^+ \wedge l^+)$$

which are natural in both variables.

THEOREM 1·1. ([**Ly**, theorem 2·18]). *The smash product of Γ-spaces is associative and commutative with unit $\mathbb{S}$, up to coherent natural isomorphism. There is a natural isomorphism of Γ-spaces*

$$\mathrm{Hom}\,(X \wedge Y, Z) \cong \mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z)).$$

In other words, the category of Γ-spaces becomes a symmetric monoidal closed category.

A Γ-space X is called *special* if the map $X((k+l)^+) \to X(k^+) \times X(l^+)$ induced by the projections from $(k+l)^+ \cong k^+ \vee l^+$ to k^+ and l^+ is a weak equivalence for all k and l. In this case, the weak map

$$X(1^+) \times X(1^+) \xleftarrow{\;\sim\;} X(2^+) \xrightarrow{\;X(\nabla)\;} X(1^+)$$

induces an abelian monoid structure on $\pi_0\,(X(1^+))$. Here $\nabla \colon 2^+ \to 1^+$ is defined by $\nabla(1) = 1 = \nabla(2)$. X is called *very special* if it is special and the monoid $\pi_0\,(X(1^+))$ is a group. By Segal's theorem ([**Se**, proposition 1·4] or [**BF**, theorem 4·2]), the spectrum associated to a very special Γ-space X is an Ω-spectrum in the sense that the maps $|X(S^n)| \to \Omega|X(S^{n+1})|$ adjoint to the spectrum structure maps are homotopy equivalences. In particular, the homotopy groups of a very special Γ-space X are naturally isomorphic to the homotopy groups of the simplicial set $X(1^+)$.

Simplicial abelian groups give rise to very special Γ-spaces via an Eilenberg–MacLane functor H. For a simplicial abelian group A, the Γ-space HA is defined by $(HA)(n^+) = A \otimes \widetilde{\mathbb{Z}}\,[n^+]$ where $\widetilde{\mathbb{Z}}\,[n^+]$ denotes the reduced free abelian group generated by the pointed set n^+. HA is very special and the associated spectrum is an Eilenberg–MacLane spectrum for A. The homotopy groups of HA are naturally isomorphic to the homotopy groups of A. The functor H embeds simplicial abelian groups as a full subcategory of $\mathscr{GS}$. A functor L in the opposite direction is defined as follows. For a Γ-space X, $L(X)$ is the cokernel of the map of simplicial abelian groups

$$(p_1)_* + (p_2)_* - \nabla_* \colon \widetilde{\mathbb{Z}}\,[X(2^+)] \longrightarrow \widetilde{\mathbb{Z}}\,[X(1^+)].$$

Here p_1 and p_2 are the two projections from 2^+ to 1^+ in Γ^{op}. The functor L is left adjoint and left inverse to H and it is compatible with the smash product of Γ-spaces. We summarize the properties of L in the next lemma. For 'nice' Γ-spaces (for those which are cofibrant in the Q-model category structure of Theorem 1·5), L represents spectrum homology, see Lemma 4·2. These formal and homotopical properties of H and L are the main input for the comparison of modules and algebras over Eilenberg–MacLane Gamma-rings with simplicial modules and algebras in Section 4. In the following lemma, $\mathrm{Hom}_{s\mathrm{Ab}}$ denotes the internal hom object in the category of simplicial abelian groups.

LEMMA 1·2. *The functor L is both left adjoint and left inverse to H and it preserves finite products. The adjunction extends to an isomorphism of Γ-spaces*

$$\mathrm{Hom}\,(X, HA) \cong H(\mathrm{Hom}_{s\mathrm{Ab}}\,(L(X), A)).$$

The functor L is strong symmetric monoidal, i.e. there are natural associative, unital and commutative isomorphisms

$$L(X) \otimes L(Y) \cong L(X \wedge Y) \quad and \quad \mathbb{Z} \cong L(\mathbb{S}).$$

The functor H is lax symmetric monoidal, i.e. there are natural associative, unital and commutative transformations

$$HA \wedge HB \longrightarrow H(A \otimes B) \quad and \quad \mathbb{S} \longrightarrow H\mathbb{Z}.$$

The adjunction map $X \to HL(X)$ induces an isomorphism on π_0 and an epimorphism on π_1.

Proof. We omit the proof that L is left adjoint to H and that $L(HA)$ is naturally isomorphic to A. The extended adjunction isomorphism then follows from the additional fact that the Γ-spaces $(HA)_{n^+\wedge}$ and $H(A^n)$ are naturally isomorphic. To see that L preserves finite products, we consider two Γ-spaces X and Y and note that the composite

$$L(X \vee Y) \longrightarrow L(X \times Y) \longrightarrow L(X) \times L(Y)$$

is an isomorphism since L preserves coproducts. So it suffices to show that the map $L(X \vee Y) \to L(X \times Y)$ is surjective. Let (x, y) denote one of the generators of the group $\widetilde{\mathbb{Z}}[X(1^+) \times Y(1^+)]$. We consider the element $(i_1(x), i_2(y))$ of $X(2^+) \times Y(2^+)$, where $i_1, i_2 \colon 1^+ \to 2^+$ are the two non-trivial morphisms in Γ^{op}. The image of $(i_1(x), i_2(y))$ under the map

$$(p_1)_* + (p_2)_* - \nabla_* \colon \widetilde{\mathbb{Z}}[X(2^+) \times Y(2^+)] \longrightarrow \widetilde{\mathbb{Z}}[X(1^+) \times Y(1^+)]$$

is the sum of generators $(x, *) + (*, y) - (x, y)$. In the quotient group $L(X \times Y)$, the image of the generator (x, y) is thus equal to the sum of the generators $(x, *)$ and $(*, y)$, so it is in the image of $L(X \vee Y)$.

A map $L(X) \otimes L(Y) \to L(X \wedge Y)$ is obtained as follows. The universal property of the smash product provides a map $X(1^+) \wedge Y(1^+) \to (X \wedge Y)(1^+ \wedge 1^+) = (X \wedge Y)(1^+)$. To this we apply the functor $\widetilde{\mathbb{Z}}$ and note that the resulting map passes to cokernels. The natural transformation obtained is then associative, commutative and unital. To show that it is an isomorphism, we can restrict our attention to Γ-sets. Everything in sight commutes with colimits, and every Γ-set is a colimit of representable Γ-spaces Γ^n, so it is enough to show that $L(\Gamma^n) \otimes L(\Gamma^m) \to L(\Gamma^n \wedge \Gamma^m)$ is an isomorphism. But this follows since $L(\Gamma^n)$ is naturally isomorphic to $\mathbb{Z}^n$ as a contravariant functor in $n^+ \in \Gamma^{\mathrm{op}}$ and because $\Gamma^n \wedge \Gamma^m$ is isomorphic to Γ^{nm} [**Ly**, proposition 2·15]. The symmetric monoidal transformation for H is obtained, by the universal property of $\wedge$, from the map

$$(HA)(n^+) \wedge (HB)(m^+) \longrightarrow (H(A \otimes B))(n^+ \wedge m^+)$$

that sends the smash product of two elements to their tensor product.

It remains to show that the map $X \to HL(X)$ is 1-connected. Denote by $\widetilde{\mathbb{Z}} \circ X$ the Γ-space which takes n^+ to the underlying simplicial set of the reduced free abelian group generated by $X(n^+)$. Since $HL(X)$ stems from a functor from Γ^{op} to simplicial abelian groups, the map in question factors over the inclusion of generators $X \to \widetilde{\mathbb{Z}} \circ X$. The Hurewicz theorem applied to the simplicial set $X(S^n)$ for large n shows that $X \to \widetilde{\mathbb{Z}} \circ X$ induces an isomorphism on π_0 and an epimorphism on π_1. We let K denote the pointwise kernel of the induced map $\widetilde{\mathbb{Z}} \circ X \to HL(X)$, viewed as a map of functors from Γ^{op} to simplicial abelian groups. When we take the short exact sequence

$$0 \longrightarrow K \longrightarrow \widetilde{\mathbb{Z}} \circ X \longrightarrow HL(X) \longrightarrow 0$$

of abelian group valued functors and evaluate it at a high dimensional simplicial sphere, we obtain a short exact sequence of simplicial abelian groups. The long exact homotopy sequences of those simplicial abelian groups in the limit assemble to a long exact sequence of homotopy groups for the Γ-spaces underlying $K, \widetilde{\mathbb{Z}} \circ X$ and $HL(X)$. So it suffices to show that $\pi_0 K$ is trivial.

We consider the map $X \vee X \to X_{2^+ \wedge}$ which is induced by the two inclusions $1^+ \to 2^+$. This map is a stable equivalence, hence so is the induced map

$$\widetilde{\mathbb{Z}} \circ (X \vee X) \to \widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}).$$

The map $(p_1)_* + (p_2)_* - \nabla_* \colon \widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}) \to \widetilde{\mathbb{Z}} \circ X$ becomes trivial when composed with the map $\widetilde{\mathbb{Z}} \circ X \to HL(X)$, so it factors over a map $\widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}) \to K$. By the definition of K, this map is surjective at 1^+, so $\pi_0(\widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}))$ surjects onto $\pi_0 K$. But the composite $\widetilde{\mathbb{Z}} \circ (X \vee X) \to \widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}) \to K$ is trivial and $\pi_0(\widetilde{\mathbb{Z}} \circ (X \vee X)) \to \pi_0(\widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge}))$ is an isomorphism, so $\pi_0(\widetilde{\mathbb{Z}} \circ (X_{2^+ \wedge})) \to \pi_0 K$ is both surjective and trivial, so $\pi_0 K$ is trivial.

LEMMA 1·3. *Let $A \to B$ be an injective map of Γ-spaces. Then there is a natural long exact sequence of homotopy groups*

$$\cdots \to \pi_n A \to \pi_n B \to \pi_n B/A \to \pi_{n-1} A \to \cdots \pi_0 B/A \to 0$$

where B/A denotes the cofibre (quotient) of B by A. Let $X \to Y$ be a map between very special Γ-spaces which is surjective on π_0. Assume also that $X \to Y$ is pointwise a fibration of simplicial sets, and let F denote its fibre. Then the induced map $X/F \to Y$ is a stable equivalence.

Proof. For every i, the sequence

$$A(S^i) \to B(S^i) \to (A/B)(S^i)$$

is a cofibre sequence of simplicial sets in which all objects are $(i-1)$-connected. Hence for these simplicial sets there is a long exact sequence of homotopy groups in a range of roughly $2i$ and the desired long exact sequence is obtained as i goes to infinity. Under the hypothesis of the second part of the lemma, the fibre F is also very special. So the homotopy groups of X, Y and F are naturally isomorphic to the homotopy groups of the simplicial sets $X(1^+), Y(1^+)$ and $F(1^+)$, for which there is a long exact sequence. By the first part and the five lemma, $X/F \to Y$ is thus a stable equivalence. $\square$

Bousfield and Friedlander introduce two model category structures for Γ-spaces called the *strict* and the *stable* model categories [**BF**, 3·5, 5·2]. It will be more convenient for our purposes to work with slightly different model category structures, which we call the Quillen- or Q-model category structures. The strict and stable Q-structures have the same weak equivalences as the corresponding Bousfield–Friedlander model category structures, but different fibrations and cofibrations. The strict Q-structure is mentioned in [**BF**] at the end of Section 3.

We call a map of Γ-spaces a *strict Q-fibration* (resp. a *strict Q-equivalence*) if it is a Kan fibration (resp. weak equivalence) of simplicial sets when evaluated at every $n^+ \in \Gamma^{\mathrm{op}}$. *Strict Q-cofibrations* are defined as the maps having the left lifting property with respect to all strict acyclic Q-fibrations. The Q-cofibrations can be characterized

in the spirit of [**Q**, II·4, remark 4] as the injective maps with free cokernel, see Lemma A3 for the precise statement. The injection $\Gamma^k \vee \Gamma^l \to \Gamma^{k+l}$ is an example of a map which is not a Q-cofibration, but which is a cofibration in the sense of [**BF**, 3·5]. On the other hand, the Eilenberg–MacLane Γ-space HA is strictly Q-fibrant for every simplicial abelian group A, but it is not strictly fibrant in the sense of [**BF**, 3·5] unless A is a constant simplicial abelian group.

The definitions can be rephrased as saying that a map of Γ-spaces is a strict Q-fibration (resp. strict Q-equivalence) if and only if it induces a fibration (resp. weak equivalence) of simplicial hom sets hom $(\Gamma^n, -)$ for all the representable Γ-spaces Γ^n. These objects Γ^n form a set of small projective generators in the sense of [**Q**, II·4, p. 4·1] for the category of Γ-sets. Since a Γ-space can be viewed as a simplicial object of Γ-sets, we have

LEMMA 1·4. [**Q**, II·4, theorem 4]. *The strict Q-notions of weak equivalences, fibrations and cofibrations make the category of* Γ-*spaces into a closed simplicial model category.*

More important is the *stable* Q-model category structure. This one is obtained by localizing the strict Q-model category structure with respect to the stable equivalences. We call a map of Γ-spaces a *stable Q-equivalence* if it induces isomorphisms on homotopy groups. The *stable Q-cofibrations* are the strict Q-cofibrations and the *stable Q-fibrations* are defined by the right lifting property with respect to the stable acyclic Q-cofibrations. The proof of the following theorem will be given in Appendix A. It relies on Bousfield's cardinality argument of [**Bou**, section 11].

THEOREM 1·5. *The stable notions of Q-cofibrations, Q-fibrations and Q-equivalences make the category of* Γ-*spaces into a cofibrantly generated closed simplicial model category. A* Γ-*space* X *is stably Q-fibrant if and only if it is very special and* $X(n^+)$ *is fibrant as a simplicial set for all* $n^+ \in \Gamma^{\mathrm{op}}$. *A strict Q-fibration between stably Q-fibrant* Γ-*spaces is a stable Q-fibration.*

An adjoint functor pair between model categories is called a *Quillen pair* if the left adjoint L preserves cofibrations and acyclic cofibrations. An equivalent condition is to demand that the right adjoint R should preserve fibrations and acyclic fibrations. Under these conditions, the functors pass to an adjoint functor pair on homotopy categories (see [**Q**, I·4, theorem 3], [**DS**, theorem 9·7(i)]). A Quillen functor pair is called a *Quillen equivalence* if the following condition holds: for every cofibrant object A of the source category of L and for every fibrant object X of the source category of R, a map $L(A) \to X$ is a weak equivalence if and only if its adjoint $A \to R(X)$ is a weak equivalence. Sometimes the right adjoint functor R preserves and detects all weak equivalences. Then the pair is a Quillen equivalence if for all cofibrant A the unit $A \to R(L(A))$ of the adjunction is a weak equivalence. A Quillen equivalence induces an equivalence of homotopy categories (see [**Q**, I·4, theorem 3], [**DS**, theorem 9·7(ii)]), but it also preserves higher order structure like (co-)fibration sequences, Toda brackets and the homotopy types of function complexes. If two model categories are related by a chain of Quillen equivalences, they can be viewed as having the same homotopy theory.

Remark 1·6. The Bousfield–Friedlander stable equivalences and the stable Q-equivalences coincide: they are the maps inducing isomorphisms on homotopy

groups. Furthermore, every Q-cofibration is a cofibration in the sense of [**BF**, 3·5, 5·2] and hence every stable fibration in the sense of [**BF**, 5·2] is a stable Q-fibration. So the two stable structures are essentially equivalent. One advantage of the Q-model category structure is that it is compatible with the adjoint functors H and L. The Eilenberg–MacLane functor H takes acyclic fibrations of simplicial abelian groups to strict acyclic Q-fibrations of Γ-spaces. Since the acyclic fibrations coincide in the strict and stable Q-structures, H takes acyclic fibrations to stable acyclic Q-fibrations. The functor H also takes fibrations of simplicial abelian groups to strict Q-fibrations between stably Q-fibrant Γ-spaces, which are stable Q-fibrations by Theorem 1·5. So H and L are a Quillen pair with respect to the stable Q-model category structure. Since the Γ-space HA is *not* fibrant in the strict Bousfield–Friedlander structure (unless the simplicial abelian group A is discrete), H and L do *not* form a Quillen pair with respect to the Bousfield–Friedlander model structure.

The following compatibility of the Q-model category structure with the smash product is an easy consequence of Lydakis' results [**Ly**]. It can be rephrased as saying that Lydakis' smash product and the stable Q-model category structure make the category of Γ-spaces into a monoidal model category satisfying the monoid axiom in the sense of [**SS**, definitions 3·1, 3·3]. These are the homotopical ingredients that will enable us to lift the model category structure from Γ-spaces to modules and algebras in Section 2.

LEMMA 1·7. *Smashing with a Q-cofibrant Γ-space preserves stable equivalences.*

(Pushout product axiom) *If $A \to B$ and $K \to L$ are Q-cofibrations of Γ-spaces then so is the canonical map (called pushout product)*

$$A \wedge L \cup_{A \wedge K} B \wedge K \ \longrightarrow \ B \wedge L.$$

If in addition one of the former maps is a stable equivalence, then so is the pushout product.

(Monoid axiom) *Let I denote the smallest class of maps of Γ-spaces which contains the maps of the form $A \wedge Z \to B \wedge Z$ for $A \to B$ a stable acyclic Q-cofibration, Z any Γ-space, and which is closed under cobase change and transfinite composition. Then every map in I is a stable equivalence.*

Proof. The first statement follows from the stronger result [**Ly**, theorem 5·12] since every Q-cofibration is a cofibration in the sense of [**Ly**, definition 3·1]. If the maps $A \to B$ and $K \to L$ are Q-cofibrations, they are injective and so their pushout product is injective by [**Ly**, proposition 4·4]. By the characterization of Q-cofibrations (Lemma A3) it is thus enough to show that the cofibre of the pushout product is Q-cofibrant. Again by Lemma A3, this boils down to showing that the smash product of free Γ-sets is free. But this holds because $\Gamma^n \wedge \Gamma^m$ is isomorphic to Γ^{nm}, see [**Ly**, proposition 2·15].

Suppose $A \to B$, say, is also a stable equivalence. To show that then the pushout product is a stable equivalence, it suffices by Lemma 1·3 to show that its cofibre is stably contractible. But this cofibre is isomorphic to $(B/A) \wedge (L/K)$, which is stably contractible since smashing with the Q-cofibrant Γ-space L/K preserves stable equivalences.

It remains to prove the monoid axiom. Every map of the form $A \wedge Z \to B \wedge Z$

with A, B, Z as above is injective, and we claim that its cofibre $(B/A) \wedge Z$ is stably contractible. To see this we choose a stably equivalent Q-cofibrant replacement $Z^c \to Z$. Since smashing with the Q-cofibrant Γ-spaces B/A and Z^c preserves stable equivalence, the Γ-space $(B/A) \wedge Z$ is stably equivalent to $(B/A) \wedge Z^c$ which in turn is stably contractible since B/A is. By Lemma 1·3 the map $A \wedge Z \to B \wedge Z$ is thus a stable equivalence. Again by Lemma 1·3 and the five lemma every cobase change of an injective stable equivalence is another injective stable equivalence. Since homotopy groups commute with filtered colimits over injective maps, every transfinite composition of injective stable equivalences is another injective stable equivalence. Hence every map in I is an injective stable equivalence.

Remark 1·8. The smash product also satisfies the pushout product axiom and monoid axiom for the Bousfield–Friedlander stable model category structure and for the two strict structures, but these are of less importance for the present paper. Smashing with a Γ-space which is cofibrant in the sense of Bousfield and Friedlander (but not necessarily Q-cofibrant) also preserves stable equivalences by [**Ly**, theorem 5·12]. In all four model category structures the maps in the class I considered in the monoid axiom are injective equivalences, but they are not in general acyclic cofibrations.

2. *Gamma-rings and their modules*

Our version of ring spectra are the monoids in the symmetric monoidal category of Γ-spaces with respect to the smash product, which we call Gamma-rings. Many results of this section are formal in the sense that they follow from the existence of and a few homotopical properties of the smash product that were summarized in Lemma 1·7. A formally very similar situation arose in the study of modules and algebras in the framework of symmetric spectra [**HSS**]. We thus decided to take an axiomatic approach to the problem in [**SS**], introducing the notion of a monoidal model category satisfying the monoid axiom [**SS**, definitions 3·1, 3·3]. Hence for the following construction of the model category structures, most of the work is done in [**SS**]. In this section we apply the main results of [**SS**] to Γ-spaces and point out the special features of the Γ-space category.

Definition 2·1. A *Gamma-ring* is a monoid in the symmetric monoidal category of Γ-spaces with respect to the smash product. Explicitly, a Gamma-ring is a Γ-space R equipped with maps

$$\mathbb{S} \to R \quad \text{and} \quad R \wedge R \to R,$$

called the *unit* and *multiplication* map, which satisfy certain associativity and unit conditions (see [**MacL**, VII·3]). A Gamma-ring R is *commutative* if the multiplication map is unchanged when composed with the twist, or the symmetry isomorphism, of $R \wedge R$. A map of Gamma-rings is a map of Γ-spaces commuting with the multiplication and unit maps. If R is a Gamma-ring, a *left R-module* is a Γ-space N together with an action map $R \wedge N \to N$ satisfying associativity and unit conditions (see again [**MacL**, VII·4]). A map of left R-modules is a map of Γ-spaces commuting with the action of R. We denote the category of left R-modules by R-mod.

One similarly defines right modules. The unit $\mathbb{S}$ of the smash product is a Gamma-ring in a unique way. The category of $\mathbb{S}$-modules is isomorphic to the category of Γ-spaces. For a Gamma-ring R the opposite Gamma-ring R^{op} is defined by composing the multiplication with the twist map $R \wedge R \to R \wedge R$. Then the category of right R-modules is isomorphic to the category of left R^{op}-modules. The smash product of two Gamma-rings is naturally a Gamma-ring. An R-T-bimodule is defined to be a left $(R \wedge T^{\mathrm{op}})$-module.

In the definition of a Gamma-ring we view the multiplication map as defined on the internal smash product. Because of the universal property of the smash product, one can instead define a Gamma-ring via maps

$$n^+ = \mathbb{S}\,(n^+) \; \to \; R(n^+) \quad \text{and} \quad R(n^+) \wedge R(m^+) \; \to \; R(n^+ \wedge m^+)$$

which are natural in $n^+, m^+ \in \Gamma^{\mathrm{op}}$. The associativity and unit conditions can similarly be expanded into external form. This rewriting shows that a Gamma-ring has the same kind of structure as an FSP ('Functor with Smash Product' [**Bö**, 1·1], [**PW**, 2·2]). The main difference is that FSPs are indexed on all finite pointed simplicial sets, whereas Gamma-rings are indexed on finite discrete sets. This justifies thinking of Gamma-rings as 'FSPs defined on finite sets'. A similar, but different, specialization of FSPs leads to the symmetric ring spectra in the sense of J. Smith. These can be viewed as 'FSPs defined on spheres' because they contain the information one obtains from an FSP by forgetting everything except its value at the simplicial spheres S^n and their coordinate permutations. The two notions of Gamma-rings and symmetric ring spectra are in a sense 'orthogonal' because the indexing categories used (Γ^{op} versus spheres and coordinate permutations) only intersect in $1^+ \cong S^0$.

Standard examples of Gamma-rings are monoid rings over the sphere Gamma-ring $\mathbb{S}$ and Eilenberg–MacLane models of classical rings. Both these kinds of Gamma-rings are in fact the restrictions to finite sets of the standard FSP models (cf. [**BHM**, 3·2], [**PW**, 2·3]). If M is a simplicial monoid, we define a Γ-space $\mathbb{S}[M]$ by

$$\mathbb{S}[M]\,(n^+) = M^+ \wedge n^+.$$

So $\mathbb{S}[M]$ is isomorphic to $\mathbb{S} \wedge M^+$ and it represents the suspension spectrum of M^+. There is a unit map $\mathbb{S} \to \mathbb{S}[M]$ induced by the unit of M and a multiplication map $\mathbb{S}[M] \wedge \mathbb{S}[M] \to \mathbb{S}[M]$ induced by the multiplication of M which turn $\mathbb{S}[M]$ into a Gamma-ring. This construction of the monoid ring over $\mathbb{S}$ is left adjoint to the functor which takes a Gamma-ring R to the simplicial monoid $R(1^+)$.

If B is a simplicial ring, then the Eilenberg–MacLane Γ-space HB is naturally a Gamma-ring, simply because H is a lax monoidal functor. The functor H is still full and faithful when considered as a functor from the category of simplicial rings to the category of Gamma-rings. The functor L is still left adjoint and left inverse to H.

A formal consequence of having a closed symmetric monoidal smash product is that the category of R-modules inherits a smash product and function objects. More precisely, let M be a right R-module, N a left R-module. Define the smash product $M \wedge_R N$ as the coequalizer, in the category of Γ-spaces, of the two maps $M \wedge R \wedge N \rightrightarrows M \wedge N$ given by the action of R on M and N respectively. If M happens to be a T-R-bimodule and N an R-U-bimodule, then $M \wedge_R N$ is naturally a T-U-bimodule. In particular, if R is a commutative Gamma-ring, the notions of left

and right module coincide and agree with the notion of a symmetric bimodule. In this case $\wedge_R$ is an internal symmetric monoidal smash product for R-modules.

The analogous phenomenon happens for internal function objects. Given two left R-modules M and N, there are two maps of Γ-spaces $\mathrm{Hom}\,(M,N) \rightrightarrows \mathrm{Hom}\,(R \wedge M, N)$. The first is induced by the action map for M, the second is the composition of $R \wedge - \colon \mathrm{Hom}\,(M,N) \to \mathrm{Hom}\,(R \wedge M, R \wedge N)$ followed by the map induced by the action of R on N. We define the function Γ-space $\mathrm{Hom}_R\,(M,N)$ of R-morphisms between M and N to be the equalizer of these two maps. As was the case with the smash product, the usual properties known from algebra carry over: if M is an R-T-bimodule and N an R-U-bimodule, then $\mathrm{Hom}_R\,(M,N)$ is naturally a T-U-bimodule. In particular, over a commutative Gamma-ring we get internal function objects for the module category.

Finally, the adjunction between smash and function Γ-spaces extends as follows: consider a left R-module N, a T-R-bimodule M and a left T-module W. Then there is an isomorphism of Γ-spaces, natural in all three variables

$$\mathrm{Hom}_T\,(M \wedge_R N, W) \;\cong\; \mathrm{Hom}_R\,(N, \mathrm{Hom}_T\,(M, W)).$$

In particular, over a commutative Gamma-ring the internal smash and function objects are again adjoint.

We now establish the model category structure for modules over a Gamma-ring R. It is created by the forgetful functor R-mod$\to \mathscr{GS}$. By this we mean that a map of R-modules is called a *weak equivalence* (resp. *fibration*) if it is a stable Q-equivalence (resp. stable Q-fibration) as a map of Γ-spaces. A map of R-modules will be called a *cofibration* if it has the left lifting property with respect to all acyclic fibrations in R-mod. Theorem 4·4 will show that for a simplicial ring B, the homotopy theory of HB-modules is equivalent to that of simplicial B-modules.

THEOREM 2·2. *For any Gamma-ring R, the category of left R-modules is a cofibrantly generated closed simplicial model category. Every cofibration of R-modules is injective. If N is a cofibrant left R-module then the functor $- \wedge_R N$ takes stable equivalences of right R-modules to stable equivalences of* Γ-*spaces.*

Proof. Lemma 1·7 states that the smash product makes the category of Γ-spaces into a monoidal model category satisfying the monoid axiom in the sense of [**SS**, definitions 3·1, 3·3]. The model category structure is cofibrantly generated by Theorem 1·5 and every Γ-space is small with respect to the whole category since the category of Γ-spaces is locally finitely presentable (see Appendix A). So [**SS**, theorem 4·1(1)] shows that R-modules form a cofibrantly generated model category. The simplicial structure, i.e. mapping objects from and products with simplicial sets, is defined on the underlying Γ-spaces and the simplicial axiom SM7 holds for R-modules since it holds for Γ-spaces. Every generating cofibration is injective and injectivity is closed under pushouts, transfinite composition and retracts. By the small object argument (Lemma A1), every cofibration of R-modules can be obtained using these operations.

Call a left R-module N *nice* if smashing over R with N preserves stable equivalences. It remains to show that cofibrant modules are nice. Consider a Q-cofibration of Γ-spaces $A \to B$ and a diagram of R-modules

$$N \;\longleftarrow\; R \wedge A \;\longrightarrow\; R \wedge B$$

and suppose that N is nice. We claim that then the pushout of this diagram is also nice. To see this, we smash the pushout square with a weak equivalence to obtain a certain cube of Γ-spaces. Viewed as a map between pushout squares, all horizontal maps in this cube are injective. This provides long exact sequences of homotopy groups (Lemma 1·3) and the claim follows by the five lemma. Since homotopy groups commute with filtered colimits over injective maps and smashing over R commutes with colimits and takes cofibrations to injective maps, the class of nice modules is closed under transfinite composition along cofibrations. By the small object argument (Lemma A1), every cofibrant left R-module can be obtained from the trivial module by pushouts and transfinite composition as above, and retracts. Hence every cofibrant R-module is nice. $\quad\square$

For a map $R \to R'$ of Gamma-rings, there is a Quillen adjoint functor pair analogous to restriction and extension of scalars: any R'-module becomes an R-module if we let R act through the map. This functor has a left adjoint taking an R-module M to the R'-module $R' \wedge_R M$. If $R \to R'$ is a stable equivalence, then for every cofibrant R-module M, the map $M \cong R \wedge_R M \to R' \wedge_R M$ is a stable equivalence. So we have

COROLLARY 2·3. *For a weak equivalence of Gamma-rings $R \xrightarrow{\sim} R'$, the functors of restriction and extension of scalars are a Quillen equivalences between the categories of R-modules and R'-modules.*

Gamma-rings form a model category with fibrations and weak equivalences defined on the underlying Γ-spaces. Gamma-rings are the same as $\mathbb{S}$-algebras and more generally algebras over any commutative Gamma-ring form a model category.

Definition 2·4. Let k be a commutative Gamma-ring. A *k-algebra* is a monoid in the symmetric monoidal category of k-modules with respect to smash product over k. Explicitly, a k-algebra is a k-module A together with associative and unital k-module maps

$$k \ \to \ A \quad \text{and} \quad A \wedge_k A \ \to A.$$

A map of k-algebras is a map of k-modules which commutes with the unit and multiplication map.

It is a formal property of symmetric monoidal categories (compare [**EKMM**, VII, 1·3]) that specifying a k-algebra structure on a Γ-space A is the same as giving A a Gamma-ring structure together with a central map of Gamma-rings $f \colon k \to A$. Here central means that the following diagram commutes

$$
\begin{array}{ccccc}
k \wedge A & \xrightarrow{\ \text{twist}\ } & A \wedge k & \xrightarrow{\ \text{id} \wedge f\ } & A \wedge A \\
{\scriptstyle f \wedge \text{id}}\downarrow & & & & \downarrow{\scriptstyle \text{mult.}} \\
A \wedge A & & \xrightarrow{\qquad \text{mult.} \qquad} & & A
\end{array}
$$

We call a map of k-algebras a *fibration* or *weak equivalence* if it is a fibration or weak equivalence on underlying k-modules (equivalently: a stable Q-fibration or stable Q-equivalence on underlying Γ-spaces). A map of k-algebras is a *cofibration* if it has the left lifting property with respect to all acyclic fibrations. Since we know

that Γ-spaces form a cofibrantly generated monoidal model category which satisfies the monoid axiom, that every Γ-space is small and that the unit $\mathbb{S}$ of the smash product is cofibrant, the following theorem is an application of [**SS**, theorem 4·1(3)]. We will also see (Theorem 4·5) that the algebras over the Eilenberg–MacLane model of a commutative simplicial ring B have the same homotopy theory as simplicial B-algebras.

THEOREM 2·5. *The category of k-algebras is a closed simplicial model category. Every cofibrant k-algebra is also cofibrant as a k-module. In particular, the category of Gamma-rings is a cofibrantly generated closed simplicial model category.*

Remark 2·6. A concluding remark about commutativity of multiplication for Gamma-rings is in order. One can consider the full subcategory of the commutative Gamma-rings. This category of commutative Gamma-rings is definitely *not* a model category with weak equivalences and fibrations defined on underlying Γ-spaces. If commutative Gamma-rings became a model category this way, the sphere Gamma-ring $\mathbb{S}$ would have a weakly equivalent fibrant model inside this category. Evaluating this fibrant representative on $1^+ \in \Gamma^{\mathrm{op}}$ would give a commutative simplicial monoid weakly equivalent to QS^0. This would imply that the space QS^0 is weakly equivalent to a product of Eilenberg–MacLane spaces, which is not the case.

3. *The derived smash product*

For a Gamma-ring R, we define the derived smash product of a right R-module M and a left R-module N in the usual way: we choose a cofibrant left R-module N^{c} and a weak equivalence $N^{\mathrm{c}} \xrightarrow{\sim} N$ and set $M \wedge_R^L N = M \wedge_R N^{\mathrm{c}}$. It follows from the last statement in Theorem 2·2 that the derived smash product is well defined up to stable equivalence and that we could have chosen to cofibrantly resolve the right module M instead. The derived smash product can be made functorial by a functorial choice of cofibrant approximation using the small object argument (Lemma A1).

There are standard spectral sequences for calculating the homotopy of $M \wedge_R^L N$, analogous to Quillen's spectral sequences for the derived tensor product of simplicial modules [**Q**, II·6, theorem 6].

LEMMA 3·1. *Let R be a Gamma-ring, M a right R-module and N a left R-module. There are functorial convergent first quadrant spectral sequences with differential of standard homological form*

$$\mathrm{E}_{p,q}^2 = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q \;\; \Longrightarrow \;\; \pi_{p+q}(M \wedge_R^L N)$$

$$\mathrm{E}_{p,q}^2 = \pi_p((H\pi_q M) \wedge_R^L N) \;\; \Longrightarrow \;\; \pi_{p+q}(M \wedge_R^L N)$$

$$\mathrm{E}_{p,q}^2 = \pi_p(M \wedge_R^L (H\pi_q N)) \;\; \Longrightarrow \;\; \pi_{p+q}(M \wedge_R^L N).$$

Proof. By functorial replacement we can assume that M is fibrant and N is cofibrant. For the construction of the first spectral sequence, we denote by $F(M)$ the wedge

$$F(M) = \bigvee_{\Sigma^n R \to M} \Sigma^n R$$

of copies of suspensions of R considered as a right module over itself; the copies are indexed by all R-module maps $\Sigma^n R \to M$ for varying n. Since M is fibrant it is very

special, and the associated spectrum is an $\mathbf{\Omega}$-spectrum. Thus every element in $\pi_n M$ is represented by a map of simplicial sets $S^n \to M(1^+)$. By adjointness, this gives a map of R-modules $\Sigma^n R = S^n \wedge R \to M$. On homotopy, this map takes the n-fold suspension of $1 \in \pi_0 R$ to the given element in $\pi_n M$. So the natural evaluation map $F(M) \to M$ is surjective on homotopy groups. Furthermore, $\pi_* F(M)$ is graded free over $\pi_* R$.

We set $P_0 = M$ and define P_{i+1} inductively as the homotopy fibre, in mod-R, of the map $F(P_i) \to P_i$. This means that P_{i+1} is the categorical fiber of the fibration which is part of the functorial factorization

$$F(P_i) \xrightarrow{\sim} E(P_i) \longrightarrow P_i$$

as a stable equivalence followed by a stable Q-fibration. Then P_{i+1} is fibrant, hence $F(P_{i+1}) \to P_{i+1}$ again induces epimorphisms on homotopy groups. Thus taking homotopy groups of the sequence of R-modules

$$\cdots \longrightarrow E(P_{i+1}) \longrightarrow E(P_i) \longrightarrow \cdots \longrightarrow E(P_0) \longrightarrow M$$

gives a functorial free resolution of $\pi_* M$ over $\pi_* R$. The fibration $E(P_i) \to P_i$ is surjective on π_0. So by Lemma 1·3 the cofibre $E(P_i)/P_{i+1}$ maps to P_i by a stable equivalence. Since smashing with N over R preserves colimits, weak equivalences and injective maps, the sequence

$$P_{i+1} \wedge_R N \longrightarrow E(P_i) \wedge_R N \longrightarrow P_i \wedge_R N$$

gives rise to a long exact sequence of homotopy groups. These homotopy groups thus assemble into an exact couple of standard homological form with $\mathrm{E}^1_{p,q} \cong \pi_{p+q} (E(P_p) \wedge_R N)$. Since $E(P_p) \wedge_R N$ is stably equivalent to

$$F(P_p) \wedge_R N \cong \bigvee_{\Sigma^n R \to P_p} \Sigma^n N,$$

we have $\mathrm{E}^1_{p,q} \cong (\pi_* E(P_p) \otimes_{\pi_* R} \pi_* N)_{p+q}$. Under this isomorphism, the differential d^1 is isomorphic to the map obtained by tensoring the above free resolution of $\pi_* M$ with $\pi_* N$. Thus the E^2-term of the spectral sequence is isomorphic to the Tor groups over $\pi_* R$.

Now we construct the second spectral sequence, the third being analogous. We follow Quillen's construction in [$\mathbf{Q}$, II·6, theorem 6], so we first provide connected covers for (right, say) R-modules. The category of R-modules has adjoint suspension and loop functors and they are given by pointwise suspension and loop on the underlying Γ-spaces. Fibrant R-modules are very special Γ-spaces, so for these looping shifts homotopy groups. If a module is looped beyond its connectivity, the lower homotopy groups are cut off. Also, for any right R-module M, the cofibre sequence

$$M \longrightarrow \mathrm{Cone}\,(M) \longrightarrow \Sigma M$$

induces long exact sequences of homotopy groups by Lemma 1·3. Hence suspension shifts homotopy groups for any module. These two facts together imply that for a fibrant right R-module M, the module $\Sigma^n \Omega^n M$ is $(n-1)$-connected and the counit of the adjunction $\Sigma^n \Omega^n M \to M$ induces an isomorphism on homotopy groups from dimension n upwards.

For the construction of the spectral sequence we may again assume that M is fibrant as a right R-module and that N is cofibrant as a left R-module. We denote by P_n the homotopy cofibre, in the category of right R-modules, of the map $\Sigma^{n+1}\Omega^{n+1}M \to \Sigma^n\Omega^n M$. Then P_n has only one non-trivial homotopy group isomorphic to $\pi_n M$ in dimension n. P_n is thus stably equivalent to the n-fold suspension of the module $H\pi_n M$. Smashing with the cofibrant right R-module N preserves homotopy cofibre sequences, so the homotopy groups of the homotopy cofibre sequences

$$(\Sigma^{n+1}\Omega^{n+1}M) \wedge_R N \ \longrightarrow \ (\Sigma^n\Omega^n M) \wedge_R N \ \longrightarrow \ P_n \wedge_R N$$

form an exact couple with

$$E_{p,q}^2 = \pi_{p+q}\,(P_q \wedge_R N) \cong \pi_{p+q}\,(\Sigma^q (H\pi_q M) \wedge_R^L N) \cong \pi_p\,((H\pi_q M) \wedge_R^L N).$$

The spectral sequence we are looking for is the one associated to this exact couple.

4. *Eilenberg–MacLane Gamma-rings and simplicial rings*

In this section we prove that modules and algebras over Eilenberg–MacLane Gamma-rings have the same homotopy theory as simplicial modules and algebras (Theorems 4·4 and 4·5). The comparison of $H\mathbb{Z}$-algebras and simplicial rings was motivated by a construction of Dundas. He shows in [**Du**] that certain FSPs which should be (equivalent to FSPs arising from) simplicial rings actually are so. Loosely speaking the main result of this section states that a Gamma-ring is stably equivalent to a simplicial ring if and only if it contains $H\mathbb{Z}$ in its center.

Since the Eilenberg–MacLane functor H from simplicial abelian groups to Γ-spaces is lax monoidal, it induces a functor B-mod $\to HB$-mod for any simplicial ring B. Since the left adjoint L is also monoidal, $L(R)$ is a simplicial ring for every Gamma-ring R, and L induces a functor R-mod $\to L(R)$-mod. Both functors preserve commutativity. Since $L(HB)$ is naturally isomorphic to B as a simplicial ring, H and L in particular pass to adjoint functors between the categories of simplicial B-modules and HB-modules.

LEMMA 4·1. *Let R be a Gamma-ring, M a right R-module and N a left R-module. Then there are natural isomorphisms*

$$L(M) \otimes_{L(R)} L(N) \ \cong L(M \wedge_R N) \quad \text{and} \quad \pi_0 M \otimes_{\pi_0 R} \pi_0 N \ \cong \ \pi_0(M \wedge_R N).$$

In particular if $R = HB$ is an Eilenberg–MacLane Gamma-ring, then $L(M \wedge_{HB} N)$ is naturally isomorphic to $L(M) \otimes_B L(N)$.

Proof. The smash product $M \wedge_R N$ was defined as a certain coequalizer in the category of Γ-spaces. If we apply L to the coequalizer diagram and use the properties of L summarized in Lemma 1·2, we obtain a coequalizer diagram of simplicial abelian groups

$$L(M) \otimes L(R) \otimes L(N) \ \rightrightarrows \ L(M) \otimes L(N) \ \longrightarrow \ L(M \wedge_R N).$$

So $L(M \wedge_R N)$ is naturally isomorphic to $L(M) \otimes_{L(R)} L(N)$. Since $\pi_0 X$ is naturally isomorphic to $\pi_0 L(X)$ for any Γ-space X (Lemma 1·2), the second isomorphism follows from the first.

LEMMA 4·2. *Let R be a Gamma-ring, M any simplicial right $L(R)$-module and N a cofibrant left R-module. Then the map*

$$HM \wedge_R N \longrightarrow H(M \otimes_{L(R)} L(N))$$

which is adjoint to the isomorphism $L(HM \wedge_R N) \cong M \otimes_{L(R)} L(N)$ is a stable equivalence. In particular, for every Q-cofibrant Γ-space X, the map $H\mathbb{Z} \wedge X \to HL(X)$ is a stable equivalence. So for Q-cofibrant X, the homotopy groups of $L(X)$ are the spectrum homology groups of X.

Proof. If N is of the form $N = R \wedge K$ for some pointed simplicial set K, then $L(N)$ is naturally isomorphic to the reduced free $L(R)$-module generated by K. The map in question then becomes $HM \wedge K \to H(M \otimes \widetilde{\mathbb{Z}}[K])$. This is an isomorphism for $K = S^0$ and the homotopy groups of both sides are reduced generalized homology theories in K (namely singular homology with coefficients in M). Hence the map is a stable equivalence for all K. Both sides of the original map $HM \wedge_R N \to H(M \otimes_{L(R)} L(N))$ preserves stable equivalences between cofibrant objects in the variable N. So if N is of the form $N = R \wedge \Gamma^n \wedge K$, it is equivalent to $R \wedge (n^+ \wedge K)$, and the map is a stable equivalence by the first part.

Now suppose $K \to L$ is a cofibration of pointed simplicial sets, N a left R-module and N' the pushout of a diagram of left R-modules

$$N \longleftarrow R \wedge \Gamma^n \wedge K \longrightarrow R \wedge \Gamma^n \wedge L.$$

We claim that if $HM \wedge_R N \to H(M \otimes_{L(R)} L(N))$ is a stable equivalence, then the same is true for N'. This claim follows from the five lemma using the fact that L preserves colimits and cofibrations and that cofibrations of R-modules and simplicial $L(R)$-modules give rise to long exact sequences of homotopy groups. Every cofibrant R-module can be obtained from the trivial module by pushouts of this kind, transfinite composition along cofibrations and retracts, so this proves the lemma for an arbitrary cofibrant R-module N. The claims about spectrum homology follow by taking $R = \mathbb{S}$ and $M = \mathbb{Z}$.

Remark 4·3. For a Γ-space which is cofibrant in the sense of Bousfield and Friedlander the functor L does not necessarily represent spectrum homology. As an example we consider the map $\Gamma^1 \vee \Gamma^1 \vee \Gamma^1 \to \Gamma^2$ induced by the three non-trivial maps $2^+ \to 1^+$. This map is a cofibration in the sense of [**BF**] and so its cofibre C is cofibrant in the sense of Bousfield–Friedlander. The induced map $L(\Gamma^1 \vee \Gamma^1 \vee \Gamma^1) \to L(\Gamma^2)$ is surjective, so $L(C)$ is trivial. However, the Γ-space C represents the suspension spectrum is S^1, which certainly has non-trivial spectrum homology.

For the following two comparison theorems it is important that we based the model category structures for HB-modules and HB-algebras on the Quillen-, and not the Bousfield–Friedlander model category structures for Γ-spaces (see Remark 1·6).

THEOREM 4·4. *Let B be a simplicial ring. Then the adjoint functors H and L are a Quillen equivalence between the categories of simplicial B-modules and HB-modules.*

THEOREM 4·5. *Let B be a commutative simplicial ring. Then the adjoint functors H and L are a Quillen equivalence between the categories of simplicial B-algebras and HB-algebras.*

Proofs. The fibrations and acyclic fibrations of simplicial B-modules are defined on underlying simplicial abelian groups, and the fibrations and acyclic fibrations of HB-modules are defined on underlying Γ-spaces, and similarly for algebras. So Remark 1·6 implies that the adjoint functors H and L form a Quillen pair between the model categories of simplicial B-modules and HB-modules, and similarly for algebras. The right adjoint functor H detects and preserves all weak equivalences, so to prove Theorem 4·4 it remains to shows that for every cofibrant HB-module N the unit map $N \to HL(N)$ of the adjunction is a stable equivalence. This is a special case of the previous Lemma 4·2 with $R = HB$ and $M = B$. In the case where B is commutative, every cofibrant HB-algebra is also cofibrant as an HB-module (Theorem 2·5), so the same argument takes care of Theorem 4·5.

Appendix A. The stable Q-*model category structure*

The model category structures for Gamma-rings and modules over Gamma-rings that we introduced in Section 2 are based on the stable Quillen model category structure for Γ-spaces and on the fact that this is cofibrantly generated. In this appendix we recall from [**DHK**] the definition of cofibrantly generated model categories and establish the stable Q-model structure. The main tool is Bousfield's transfinite version [**Bou**, section 11] of Quillen's small object argument.

In [**Q**, II p. 3·4], Quillen formulates his small object argument, which is now a standard device in model category theory. After Quillen, several authors have axiomatized and generalized the small object argument (see e.g. [**Bl**, definition 4·4], [**Cr**, definition 3·2] or [**Sch1**, definition 1·3·1]). For Γ-spaces the countable small object argument is not good enough (see Example A6), so we need a transfinite version. An axiomatization suitable for our purposes are the 'cofibrantly generated model categories' of [**DHK**]. If a model category is cofibrantly generated, its model category structure is determined by a set of cofibrations and a set of acyclic cofibrations. There is a functorial factorization of maps as cofibrations followed by acyclic fibrations and as acyclic cofibrations followed by fibrations. The author is grateful to Dwyer, Hirschhorn and Kan for giving him access to early drafts of their book [**DHK**].

Ordinals and cardinals. An *ordinal* γ is an ordered isomorphism class of well ordered sets. It can be identified with the well ordered set of all preceding ordinals. For an ordinal γ, the same symbol will denote the associated poset category. The latter has an initial object $\varnothing$, the empty ordinal. An ordinal is a *limit ordinal* if it has no immediate predecessor. An ordinal κ is a *cardinal* if its cardinality is bigger than that of any preceding ordinal. A cardinal κ is called *regular* if the following holds: for every set of sets $\{X_j\}_{j \in J}$ indexed by a set J of cardinality less than κ and such that the cardinality of each X_j is less than that of κ, the cardinality of the union $\bigcup_J X_j$ is also less than that of κ. The successor cardinal (the smallest cardinal of bigger cardinality) of every cardinal is regular.

Transfinite composition. Let $\mathscr{C}$ be a cocomplete category and γ a well-ordered set which we identify with its poset category. A functor $V: \gamma \to \mathscr{C}$ is called a γ-*sequence* if for every limit ordinal $\beta < \gamma$ the natural map $\mathrm{colim} V|_\beta \to V(\beta)$ is an isomorphism. The map $V(\varnothing) \to \mathrm{colim}_\gamma V$ is called the *transfinite composition* of the maps of V. A subcategory $\mathscr{C}_1 \subset \mathscr{C}$ is said to be *closed under transfinite composition* if for every

ordinal γ the following holds: given a γ-sequence $V \colon \gamma \to \mathscr{C}$ such that $V(\alpha) \to V(\alpha+1)$ is in $\mathscr{C}_1$ for every $\alpha < \gamma$, then the induced map $V(\varnothing) \to \operatorname{colim}_\gamma V$ is also in $\mathscr{C}_1$. Examples of such subcategories are the cofibrations or the acyclic cofibrations in a closed model category.

Relatively small objects. Consider a cocomplete category $\mathscr{C}$ and a subcategory $\mathscr{C}_1 \subset \mathscr{C}$ closed under transfinite composition. If κ is a regular cardinal, an object $C \in \mathscr{C}$ is called κ-*small relative to* $\mathscr{C}_1$ if for every regular cardinal $\lambda \geqslant \kappa$ and every functor $V \colon \lambda \to \mathscr{C}_1$ which is a λ-sequence in $\mathscr{C}$, the map

$$\operatorname{colim}_\lambda \operatorname{Hom}_\mathscr{C}(C, V) \to \operatorname{Hom}_\mathscr{C}(C, \operatorname{colim}_\lambda V)$$

is an isomorphism. An object $C \in \mathscr{C}$ is called *small* relative to $\mathscr{C}_1$ if there exists a regular cardinal κ such that C is κ-small relative to $\mathscr{C}_1$.

Locally presentable categories. In many cases of interest, category theory automatically takes care of smallness conditions. This applies to suitable functor categories with values in simplicial sets and it applies in particular to Γ-spaces. In these cases every object is small with respect to the whole category. The relevant category theoretical notion is that of a *locally presentable category*. In general, categories involving actual topological spaces tend *not* to be locally presentable.

An object K of a category $\mathscr{C}$ is called *finitely presentable* if the hom functor $\operatorname{hom}_\mathscr{C}(K, -)$ preserves filtered colimits. For any regular cardinal κ, a poset is called κ-*filtered* if every subset of cardinality less that κ has an upper bound. An object K of $\mathscr{C}$ is called κ-*presentable* if the hom functor $\operatorname{hom}_\mathscr{C}(K, -)$ commutes with κ-filtered colimits. Note that the poset indexing a transfinite composition of length $\geqslant \kappa$ is κ-filtered (an upper bound is given by the union of cardinals involved). So κ-presentable objects are in particular κ-small with respect to the whole category $\mathscr{C}$. A set $\mathscr{G}$ of objects of a category $\mathscr{C}$ is called a set of *strong generators* if for every object K and every proper subobject there exists $G \in \mathscr{G}$ and a morphism $G \to K$ which does not factor through the subobject. A category is called *locally finitely presentable* (resp. *locally κ-presentable*) if it is cocomplete and has a set of finitely presentable (resp. κ-presentable) strong generators. A category is called *locally presentable* if it is locally κ-presentable for some regular cardinal κ. In the category of Γ-spaces, the collection of objects $\Gamma^n \wedge (\Delta^i)^+$ for varying n and i form a set of finitely presentable strong generators. So the category of Γ-spaces if locally finitely presentable. The main property of locally presentable categories for our purpose is that every object is presentable, hence small with respect to the whole category (see [**Bor**, proposition 5·2·10]).

I-injectives, I-cofibrations and regular I-cofibrations. Given a cocomplete category $\mathscr{C}$ and a set I of maps, we denote

- by I-inj the subcategory of $\mathscr{C}$ consisting of maps which have the right lifting property with respect to the maps in I. Maps in I-inj are referred to as I-*injectives*.
- by I-cof the subcategory of $\mathscr{C}$ consisting of maps which have the left lifting property with respect to the maps in I-inj. Maps in I-cof are referred to as I-*cofibrations*.

- by I-cof$_{\mathrm{reg}} \subset I$-cof the subcategory of the (possibly transfinite) compositions of maps that can be obtained via cobase change from maps in I. Maps in I-cof$_{\mathrm{reg}}$ are referred to as *regular* I*-cofibrations*.

Quillen's small object argument [**Q**, II p. 3·4] has the following transfinite analogue:

LEMMA A1. [**DHK**]. *Let* $\mathscr{C}$ *be a cocomplete category and* I *a set of maps in* $\mathscr{C}$ *whose domains are small relative to* I*-cof*$_{\mathrm{reg}}$*. Then*

- *there is a functorial factorization of maps* $f \in \mathrm{Mor}\,(\mathscr{C})$ *as* $f = qi$ *with* $q \in I$*-inj and* $i \in I$*-cof*$_{\mathrm{reg}}$ *and thus*
- *every* I*-cofibration is a retract of a regular* I*-cofibration.*

Definition A2. [**DHK**]. A closed model category $\mathscr{C}$ is *cofibrantly generated* if it is complete and cocomplete and there exist a set of cofibrations I and a set of acyclic cofibrations J such that the fibrations are precisely the J-injectives; the acyclic fibrations are precisely the I-injectives; the domain of each map in I (resp. in J) is small relative to I-cof$_{\mathrm{reg}}$ (resp. J-cof$_{\mathrm{reg}}$). The maps in I (resp. J) will be referred to as *generating cofibrations* (resp. *generating acyclic cofibrations*).

We need a characterization of the Q-cofibrations. We call a Γ-set *free* if it isomorphic to a wedge of (possibly infinitely many) copies of Γ^ns for various n. A Γ-space can be viewed as a simplicial object of Γ-sets and thus has degeneracies in the simplicial direction. For a Γ-space A, we denote by A_i the Γ-set obtained by fixing the simplicial degree i. We also let A_i^d denote the degenerate part, i.e. the sub-Γ-set of A_i consisting of the images of the A_j for $j < i$ under the simplicial degeneracy maps. So the set $A_i^d(n^+)$ consists of the degenerate i-simplices in the simplicial set $A(n^+)$. We then have

LEMMA A3. (*a*) *A retract of a free* Γ*-set is free.*

(*b*) *A map of* Γ*-spaces* $A \to B$ *is a Q-cofibration if and only if it is injective and the* Γ*-set* $B_i/(B_i^d \bigcup_{A_i^d} A_i)$ *is free in every simplicial degree* $i \geqslant 0$*.*

Proof. (*a*) Let $F = \bigvee_{j \in J} \Gamma^{n_j}$ be a free Γ-set. Any retract of F is isomorphic to the image of an idempotent endomorphism ϵ of F. Dropping some wedge summands if necessary, we can assume that the image of ϵ intersects every wedge summand of F non-trivially (i.e. not only in the basepoint). Since ϵ is idempotent, it then has to take every wedge summand to itself.

So ϵ decomposes as a wedge of idempotent endomorphisms of all the individual wedge summands of F. We are thus reduced to showing that a retract of a single Γ^n is free. If ϵ again denotes the corresponding idempotent endomorphism, then $\epsilon(\mathrm{id}_{n^+}) \in \Gamma^n(n^+)$ has to be idempotent as a morphism in Γ^{op}. So it can be written as a composite $\epsilon(\mathrm{id}_{n^+}) = ip$ with $p : n^+ \to k^+$, $i : k^+ \to n^+$ and $pi = \mathrm{id}_{k^+}$. Then the map $\Gamma^k \to \Gamma^n$ induced by p is an isomorphism onto the image of ϵ, so that image is free.

(*b*) This is a standard argument, similar to [**Q**, II·4, remark 4], and we omit it. To be precise, the argument of [**Q**, II·4, remark 4] only gives that the Γ-sets $B_i/(B_i^d \bigcup_{A_i^d} A_i)$ are retracts of free Γ-sets, but such Γ-sets are themselves free by part (*a*) of this lemma. $\square$

We call a Γ-space X countable if the disjoint union of all simplices in all the simplicial sets $X(n^+), n^+ \in \Gamma^{\mathrm{op}}$ is countable.
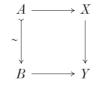
LEMMA A4. *Let A be a non-trivial, stably contractible and Q-cofibrant Γ-space. Then there exists a non-trivial, stably contractible and Q-cofibrant sub-Γ-space M which is countable.*

Proof. We adapt [**Bou**, lemma 11·2]. The desired M is obtained as the ascending union of countable, Q-cofibrant sub-Γ-spaces $M_i \subset A, i \in \mathbb{N}$. It follows from the characterization of Q-cofibrations (Lemma A3) and the fact that the categories Γ^{op} and Δ^{op} are countable that every countable sub-Γ-space of A can be enlarged to a Q-cofibrant sub-Γ-space which is still countable.

We thus choose M_0 to be any non-trivial, countable and Q-cofibrant sub-Γ-space of A. Assume that M_i has been constructed and is countable. A is the filtered union of its countable sub-Γ-spaces. Homotopy groups commute with filtered colimits over inclusions of Γ-spaces, so for any element $x \in \pi_* M_i$ there exists a countable $N_x \subset A$ such that the element x maps to 0 in the homotopy of N_x. Since M_i is countable, so are its homotopy groups, so we can define M_{i+1} to be the union of M_i with all the $N_x, x \in \pi_* M_i$. Again M_{i+1} is countable, we can assume that it is Q-cofibrant and by construction the map $M_i \to M_{i+1}$ induces the trivial map on homotopy groups. Hence if we define M as the union of the M_i, then M is countable, stably contractible and Q-cofibrant.

LEMMA A5. *Let $X \to Y$ be a map of Γ-spaces which has the right lifting property with respect to all stable acyclic Q-cofibrations between countable Γ-spaces. Then the map has the right lifting property with respect to all stable acyclic Q-cofibrations.*

Proof. We follow the proof of [**Bou**, lemma 11·3]. Let $A \to B$ be a stable acyclic Q-cofibration between arbitrarily large Γ-spaces. We can assume that it is in fact the inclusion of a sub-Γ-space, and we look at a lifting problem

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle\sim}\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

Consider the set of all $(D, f_D \colon D \to X)$ such that $A \subseteq D \subseteq B$, the inclusion $D \to B$ is a Q-cofibration and a stable equivalence and f_D is a partial lifting. The set of these (D, f_D) is non-empty and is preordered by declaring $(D, f_D) \leqslant (D', f_{D'})$ if and only if $D \subseteq D'$ and $f_{D'}$ extends f_D. Every chain in this preorder has an upper bound, so by Zorn's lemma there is a maximal element (D_0, f_{D_0}). We claim that $D_0 = B$.

If D_0 was a proper subobject of B, then by Lemma A4 we could find a countable and non-trivial M inside the cofibre B/D_0, such that M is contractible and Q-cofibrant. We let $\widetilde{M}$ denote the smallest sub-Γ-space of B which surjects onto M; this $\widetilde{M}$ is then countable as well. The inclusion $D_0 \cap \widetilde{M} \to \widetilde{M}$ is a stable acyclic Q-cofibration since its cofibre M is Q-cofibrant and stably contractible (we use Lemmas 1·3 and A3). So by assumption, $X \to Y$ has the right lifting property with respect to the inclusion $D_0 \cap \widetilde{M} \to \widetilde{M}$. Since $D_0 \to \widetilde{M} \cup D_0$ is obtained from it by cobase change, $X \to Y$ also has the right lifting property with respect to that map. This contradicts the maximality of (D_0, f_{D_0}).

Proof of Theorem 1·5. We have to show the following: the stable Q-notions of (co-)fibrations and weak equivalences make the category of Γ-spaces into a cofibrantly generated closed simplicial model category; a Γ-space X is stably Q-fibrant if and only if it is very special and $X(n^+)$ is fibrant as a simplicial set for all $n^+ \in \Gamma^{\mathrm{op}}$; a strict Q-fibration between stably Q-fibrant Γ-spaces is a stable Q-fibration.

The model category axioms CM1 (existence of limits and colimits), CM2 (saturation of stable equivalences), CM3 (closure under retracts) and one half of CM4 (lifting properties) are clear. Since the cofibrations and acyclic fibrations of the stable and the strict Q-structures coincide, the 'cofibration/acyclic fibration' part of the factorization axiom CM5 follows from the corresponding factorization in the strict Q-structure. A map is a stable acyclic fibration if and only if it is pointwise an acyclic fibration of simplicial sets. This is equivalent to having the right lifting property with respect to the Q-cofibrations

$$\Gamma^n \wedge (\partial \Delta^i)^+ \rightarrowtail \Gamma^n \wedge (\Delta^i)^+$$

for varying n, i. These maps thus form a set of generating cofibrations.

As generating acyclic cofibrations we choose a set J of representatives of the isomorphism classes of the Q-cofibrations between countable Γ-spaces which are also stable equivalences. By the small object argument A1, every map can thus be factored as a regular J-cofibration i followed by a map q which has the right lifting property with respect to all maps in J. Since every map in J is an injective stable equivalence, and since the class of injective stable equivalences is closed under cobase change and transfinite composition, the map i is a stable equivalence. Since i is also a strict Q-cofibration, it is a stable acyclic Q-cofibration. By Lemma A5, the map q is a stable Q-fibration. This completes the proof of the factorization lemma CM5.

It remains to prove the other half of CM4, i.e. that any stable acyclic Q-fibration f has the right lifting property with respect to Q-cofibrations. For this it suffices to show that f is in fact a strict acyclic Q-fibration. The small object argument in the strict Q-structure provides a factorization $f = qi$ with i a Q-cofibration and q a strict acyclic Q-fibration. In addition, i is a stable equivalence since f is. So by definition of the stable Q-fibrations, f has the right lifting property with respect to i. Thus f is a retract of the strict acyclic Q-fibration q, so it is one itself.

To prove the simplicial axiom SM7 we can equivalently show that for every Q-cofibration $A \to B$ and every cofibration of pointed simplicial sets $K \to L$, the pushout product

$$A \wedge L \bigcup\nolimits_{A \wedge K} B \wedge K \longrightarrow B \wedge L$$

is a Q-cofibration, which is a stable equivalence if one of the former maps is. This is a special case of the pushout product axiom of Lemma 1·7, which was proved without reference to the simplicial axiom SM7.

The characterization of the stably Q-fibrant Γ-spaces is obtained as follows. The map $\Gamma^k \vee \Gamma^l \to \Gamma^{k+l}$ induced by the projections from $(k+l)^+$ to k^+ and l^+ in Γ^{op} is a stable equivalence between Q-cofibrant Γ-spaces. By [**BF**, lemma 4·5], it induces a weak equivalence of simplicial hom sets $X((k+l)^+) \cong \mathrm{hom}\,(\Gamma^{k+l}, X) \to \mathrm{hom}\,(\Gamma^k \vee \Gamma^l, X) \cong X(k^+) \times X(l^+)$ for every stably Q-fibrant Γ-space X. This proves that every stably Q-fibrant X is special. The same argument applied to the map $\Gamma^1 \vee \Gamma^1 \to \Gamma^2$

induced by the projection p_1 and the fold map ∇ implies that the shearing map $\pi_0(X(1^+))^2 \to \pi_0(X(1^+))^2$ given by $(x, y) \mapsto (x, x + y)$ is an isomorphism for stably Q-fibrant X. This means that $\pi_0(X(1^+))$ is a group, so X is very special. Every stably Q-fibrant X is strictly Q-fibrant, so $X(n^+)$ is a fibrant simplicial set for all $n^+ \in \Gamma^{\mathrm{op}}$.

To prove the converse direction we assume that $X(n^+)$ is fibrant as a simplicial set for all n and that X is very special. We choose a stable acyclic Q-cofibration $X \to X^{\mathrm{f}}$ to a stably Q-fibrant Γ-space. Since both X and X^{f} are very special, the associated spectra are Ω-spectra ([**Se**, proposition 1·4] or [**BF**, theorem 4·2]). So the stable equivalence $X \to Y$ is in fact a strict Q-equivalence, thus a strict acyclic Q-cofibration between strictly Q-fibrant objects, so it has a retraction. Thus X is a retract of X^{f}, so it is stably Q-fibrant.

Finally, we consider a strict Q-fibration $f\colon X \to Y$ where X and Y are stably Q-fibrant. We have to establish the right lifting property with respect to any stable acyclic cofibration $i\colon A \to B$. Equivalently we have to show that the induced map of simplicial hom sets

$$m(i, f)\colon \hom(B, X) \longrightarrow \hom(A, X) \times_{\hom(A, Y)} \hom(B, Y)$$

is surjective on 0-simplices. Since i is a strict Q-cofibration, f is a strict Q-fibration and the strict Q-model category is simplicial, the map $m(i, f)$ is a fibration of simplicial sets. Since i is a stable acyclic Q-cofibration, X is stably Q-fibrant and the stable Q-model category is simplicial, the map $\hom(B, X) \to \hom(A, X)$ is an acyclic fibration of simplicial sets, and similarly for Y. Thus the map $m(i, f)$ is also a weak equivalence, so it is an acyclic fibration of simplicial sets, thus surjective.

Example A6. We now give an example which shows some special features of Γ-spaces from a model category point of view. It also explains why the countable small object argument is not good enough to yield the stable model category structures and why there is no explicit set of generating acyclic cofibrations.

The characterization of the stably Q-fibrant Γ-spaces in the proof of Theorem 1·5 involved certain maps which are natural candidates as generating acyclic cofibrations. The maps $\Gamma^k \vee \Gamma^l \to \Gamma^{k+l}$ induced by the projections from $k^+ \vee l^+$ to k^+ and l^+ and the map $\Gamma^1 \vee \Gamma^1 \to \Gamma^2$ induced by the projection p_1 and the fold map ∇ are stable equivalences between Q-cofibrant and finitely presentable Γ-spaces. By a simplicial mapping cylinder construction these maps can thus be turned into acyclic Q-cofibrations. The argument in the proof of Theorem 1·5 shows that a Γ-space X is stably fibrant if and only if the map $X \to *$ has the right lifting property with respect to the pushout products of all these mapping cylinder inclusions with the boundary inclusions $(\partial\Delta^i)^+ \to (\Delta^i)^+$. So stably Q-fibrant objects are detected by lifting properties with respect to an explicit set of acyclic cofibrations between finitely presentable Γ-spaces. One might hope that these maps detect stable Q-fibrations in general, but we show by an example that this is not the case. Proposition A7 provides a map of Γ-spaces which has the right lifting property with respect to all stable equivalences between finitely presentable Γ-spaces, but which is not a stable Q-fibration (hence also not a stable fibration in the sense of [**BF**, 5·2]).

To construct the example we recall that the Eilenberg–MacLane Γ-space construction generalizes to abelian monoids. A simplicial abelian monoid A gives rise to a special Γ-spaces $\widetilde{A}$ such that $\widetilde{A}(n^+) = A^n$. For a morphism $f\colon n^+ \to m^+$ in Γ^{op} the

induced map $f_*\colon A^n \to A^m$ is given by the formula

$$f_*(a_1, \ldots, a_n) = \Big(\sum_{f(j)=1} a_j, \ldots, \sum_{f(j)=m} a_j \Big).$$

If A is a simplicial abelian group, then $\widetilde{A}$ is naturally isomorphic to HA. We denote by UA the universal abelian group generated by A, so that U is left adjoint to the inclusion of simplicial abelian groups into simplicial abelian monoids. Group completion induces a stable equivalence $\widetilde{A} \to \widetilde{UA} \cong HUA$ (since the map $\widetilde{A}(S^n) \to \widetilde{UA}(S^n)$ is a weak equivalence for all $n \geqslant 1$ by [**Sp**, corollary 5·7]).

PROPOSITION A7. *Let M be the abelian monoid generated by countably many elements ω_i, $i \geqslant 0$, subject to the relations $\omega_i + \omega_{i+1} = \omega_{i+1}$. Then the map $* \to \widetilde{M}$ from the trivial* Γ-*space to $\widetilde{M}$ has the right lifting property with respect to all maps between finitely presentable* Γ-*spaces which induce an epimorphism on π_0. On the other hand, the map $* \to \widetilde{M}$ is not a stable fibration.*

We start by deriving the relevant monoid theoretic properties of M:

LEMMA A8. *The abelian monoid M is non-trivial and its group completion UM is trivial. If N is a finitely generated submonoid of M whose group completion UN is trivial, then $N = 0$.*

Proof. The assignment $\omega_i \mapsto 0$ extends to a surjective homomorphism from M to the multiplicative monoid $\{0, 1\}$ of the field with two elements. So M is nontrivial. Because of the relations $\omega_i + \omega_{i+1} = \omega_{i+1}$ all generators of M become trivial in UM, so UM is trivial. The presentation of M also implies that every element of M is a multiple of one of the generators, i.e. of the form $n \cdot \omega_i$ for some $n, i \geqslant 0$. So for every non-trivial finitely generated submonoid N there exists a maximal number j such that N contains a positive multiple $n \cdot \omega_j$ of the generator ω_j. Since j is maximal with this property, N does not contain any element y which satisfies $(n \cdot \omega_j) + y = y$, which implies that $n \cdot \omega_j$ is non-trivial in the group completion UN.

Proof of Proposition A7 Since $UM = 0$, the map $* \to \widetilde{M}$ is a stable equivalence, so if it also were a stable fibration in the Quillen or the Bousfield–Friedlander model structure, then it would be a strict equivalence. This would contradict the fact that M is non-trivial.

To show that the map $* \to \widetilde{M}$ has the right lifting property with respect to all maps between finitely presentable Γ-spaces which induce an epimorphism on π_0 it suffices to show that the homomorphism set $\mathscr{GS}(X, \widetilde{M})$ is trivial for any finitely presentable Γ-space X with $\pi_0 X = 0$. The functor $A \mapsto \widetilde{A}$ from simplicial abelian monoids to Γ-spaces has a left adjoint, which we denote τ. The value of this adjoint on a Γ-space X can be described explicitly as the coequalizer of the maps $(p_1)_* + (p_2)_*$ and ∇_* from the reduced free abelian monoid generated by $X(2^+)$ to the reduced free abelian monoid generated by $X(1^+)$. So τ is defined in the same way as the adjoint L to the Eilenberg–MacLane functor H, only with free abelian groups replaced by free abelian monoids. Note that L factors as $LX \cong U\tau X$. Since the functor $A \mapsto \widetilde{A}$ preserves filtered colimits, its left adjoint τ preserves finite presentability. In particular $\pi_0(\tau X)$ is finitely generated as a monoid. Furthermore $U\pi_0(\tau X) \cong \pi_0 U\tau X \cong \pi_0 LX \cong \pi_0 X = 0$, so every monoid homomorphism from $\pi_0(\tau X)$ to M is trivial because

M has no non-trivial finitely generated submonoids with trivial group completion. Hence by adjointness every Γ-space map from X to $\widetilde{M}$ is trivial.

Appendix B. Topological Γ-spaces

We conclude this paper with an appendix that shows that the stable Q-model category structure for Gamma-spaces has a counterpart if simplicial sets are replaced by actual topological spaces. For definiteness, by a topological space we mean a compactly generated weak Hausdorff space [**MC**, section 2]. But the arguments of this section apply more generally to any reasonable simplicial model category of topological spaces. We define a topological Γ-space to be a pointed functor from Γ^{op} to the category of pointed topological spaces and denote by $\mathscr{GT}$ the category of topological Γ-spaces. The geometric realization functor and the singular complex functor can be applied objectwise to Γ-objects and give an adjoint functor pair

$$\mathscr{GS} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathscr{GT}$$

We call a map $f : X \to Y$ in $\mathscr{GT}$ a stable equivalence (resp. stable fibration) if the map $\mathrm{Sing}\,(f) \colon \mathrm{Sing}\,X \to \mathrm{Sing}\,Y$ is a stable equivalence or stable Q-fibration in $\mathscr{GS}$ respectively. Note that a topological Γ-space X is stably fibrant if and only if $\mathrm{Sing}\,X$ is very special. A map is a stable cofibration if it has the right lifting property with respect to all stable acyclic fibrations.

THEOREM B1. *The stable notions of cofibrations, fibrations and weak equivalences make the category of topological Γ-spaces into a closed simplicial model category. The singular complex and geometric realization functors induce a Quillen equivalence between the stable Q-model category of Γ-spaces and the stable model category of topological Γ-spaces.*

The category of compactly generated weak Hausdorff spaces is cartesian closed and the category of topological Γ-spaces is tensored and cotensored over the category of pointed spaces. As for simplicial set, the smash product $X \wedge K$ of a topological Γ-space X and a pointed space K is defined by $(X \wedge K)(n^+) = X(n^+) \wedge K$. The topological space $\hom_{\mathscr{GT}}(X, Y)$ of maps between two topological Γ-spaces X and Y is given by the end

$$\int_{n^+ \in \Gamma^{\mathrm{op}}} \mathrm{map}(X(n^+), Y(n^+)),$$

where 'map' refers to the topological mapping space. Internal function objects and a smash product for topological Γ-spaces can then be defined in the same fashion as for Γ-spaces. The smash product of topological Γ-spaces satisfies the pushout product axiom (see Lemma 1·7). Since the category $\mathscr{GT}$ is (co-)tensored over spaces, it can also be tensored over the category of simplicial sets using the geometric realization and singular complex functors.

To prove Theorem B1 we need a lifting lemma for model category structures. We make no claim to originality for this lemma, as various other lifting lemmas can be found in the model category literature. Let $\mathscr{C}$ be a cofibrantly generated simplicial model category and let $\mathscr{D}$ be a complete and cocomplete category which is tensored

and cotensored over the category of simplicial sets. Consider a simplicial adjoint functor pair

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$$

By this we mean that L and R have the structure of simplicial functors and the adjunction extends to an isomorphism of simplicial hom sets. We call a map $f\colon X \to Y$ in $\mathscr{D}$ a weak equivalence (resp. fibration) if the map $R(f)\colon R(X) \to R(Y)$ is a weak equivalence (resp. fibration) in $\mathscr{C}$. A map in $\mathscr{D}$ is called a cofibration if it has the left lifting property with respect to all acyclic fibrations.

LEMMA B2. *In the above situation, assume that the following conditions hold.*

(1) *There exist a set I of generating cofibrations and a set J of generating acyclic cofibrations for the model category structure of $\mathscr{C}$ with the following property. Let $I_{\mathscr{D}}$ and $J_{\mathscr{D}}$ denote the images of the sets I and J under the left adjoint L. Then the domains of the maps in $I_{\mathscr{D}}$ and $J_{\mathscr{D}}$ are small with respect to $I_{\mathscr{D}}\text{-cof}_{\mathrm{reg}}$ and $J_{\mathscr{D}}\text{-cof}_{\mathrm{reg}}$ respectively.*

(2) *There exists a functor $Q\colon \mathscr{D} \to \mathscr{D}$ and a natural weak equivalence $X \to QX$ such that QX is fibrant for all X.*

Then the category $\mathscr{D}$ becomes a cofibrantly generated closed simplicial model category.

Remark B3. Note that the smallness condition (1) of the lifting lemma is automatically satisfied if the model category $\mathscr{C}$ is cofibrantly generated and the category $\mathscr{D}$ is locally presentable (see App. A) because then every object of $\mathscr{D}$ is small with respect to the whole category [**Bor**, proposition 5·2·10].

Proof. We use the numbering of the model category axioms as given in [**BF**, 1·1]. Limits and colimits exist in $\mathscr{D}$ by assumption. Model category axioms CM2 (saturation) and CM3 (closure properties under retracts) are clear. One half of CM4 (lifting properties) holds by definition of cofibrations in $\mathscr{D}$. The axiom SM7a that relates the model category structure to the simplicial structure holds because the right adjoint R preserves weak equivalences, fibrations, pullbacks and mapping objects from simplicial sets. So SM7a holds in $\mathscr{D}$ since it holds in $\mathscr{C}$.

The proof of the remaining axioms uses the transfinite small object argument (Lemma A1). Hypothesis (1) makes sure that the small object argument can be applied to the sets of maps $I_{\mathscr{D}}$ and $J_{\mathscr{D}}$. We start with the factorization axiom CM5. Every map in $I_{\mathscr{D}}$ is a cofibration in $\mathscr{D}$ by adjointness. Hence all $I_{\mathscr{D}}$-cofibrations are cofibrations in $\mathscr{D}$. Since I is a generating set of cofibrations for $\mathscr{C}$, the $I_{\mathscr{D}}$-injectives are precisely the acyclic fibrations in $\mathscr{D}$, again by adjointness. Hence the small object argument applied to the set $I_{\mathscr{D}}$ gives a (functorial) factorization of maps in $\mathscr{D}$ as cofibrations followed by acyclic fibrations.
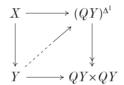
The other half of the factorization axiom CM5 needs the functor Q. Applying the small object argument to the set of maps $J_{\mathscr{D}}$ gives a functorial factorization of maps in $\mathscr{D}$ as regular $J_{\mathscr{D}}$-cofibrations followed by $J_{\mathscr{D}}$-injectives. Since J is a generating set for the acyclic cofibrations in $\mathscr{C}$, the $J_{\mathscr{D}}$-injectives are precisely the fibrations in $\mathscr{D}$, one more time by adjointness. We now adapt the argument of [**Q**, II p. 4·9] to show that every $J_{\mathscr{D}}$-cofibration $i\colon X \to Y$ is a weak equivalence. The $J_{\mathscr{D}}$-cofibrations are

precisely the maps which have the right lifting property with respect to all fibrations in $\mathscr{D}$. Since QX is fibrant, the weak equivalence $X \to QX$ can be factored through Y by a lifting

$$
\begin{array}{ccc}
X & \xrightarrow{\simeq} & QX \\
{\scriptstyle i}\downarrow & \nearrow{\scriptstyle r} & \downarrow \\
Y & \longrightarrow & *
\end{array}
$$

Then $r \circ i$ is a weak equivalence. We will show that $(Qi) \circ r$ also is a weak equivalence. Since maps in $\mathscr{C}$ are weak equivalences if and only if they become isomorphisms in the homotopy category of $\mathscr{C}$, this will prove that i is a weak equivalence.

Since QY is fibrant and the right adjoint functor R preserves the simplicial structure, the boundary inclusion $\partial\Delta^1 \to \Delta^1$ induces a fibration $(QY)^{\Delta^1} \to QY \times QY$. The map $(Qi) \circ r$ and the weak equivalence $Y \xrightarrow{\simeq} QY$ together give a map $Y \to QY \times QY$. Since i is a $J_{\mathscr{D}}$-cofibration, a lifting exists in the square

$$
\begin{array}{ccc}
X & \longrightarrow & (QY)^{\Delta^1} \\
\downarrow & \nearrow & \downarrow \\
Y & \longrightarrow & QY \times QY
\end{array}
$$

This shows that $R((Qi) \circ r)$ and the weak equivalence $R(Y) \to R(QY)$ descend to the same map in the homotopy category of $\mathscr{C}$, hence $(Qi) \circ r$ also is a weak equivalence. This finishes the proof of the factorization axiom CM5.

It remains to prove the other half of CM4, i.e. that any acyclic cofibration $A \to B$ has the left lifting property with respect to fibrations. In other words: the acyclic cofibrations are contained in the $J_{\mathscr{D}}$-cofibrations. The small object argument provides a factorization

$$
A \stackrel{\sim}{\rightarrowtail} W \twoheadrightarrow B
$$

with $A \to W$ a $J_{\mathscr{D}}$-cofibration and $W \to B$ a fibration. In addition, $W \to B$ is a weak equivalence since $A \to B$ is. Since $A \to B$ is a cofibration, a lifting in
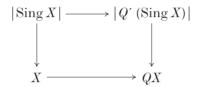
$$
\begin{array}{ccc}
A & \longrightarrow & W \\
\downarrow & \nearrow & \downarrow{\scriptstyle \sim} \\
B & \xrightarrow{=} & B
\end{array}
$$

exists. Thus $A \to B$ is a retract of a $J_{\mathscr{D}}$-cofibration, so it is one itself. $\qquad\square$

In order to deduce Theorem B1 from Lemma B2 we use the adjoint functor pair between $\mathscr{GS}$ and $\mathscr{GT}$ given by objectwise application of the singular complex and geometric realization functor. As generating cofibrations and acyclic cofibrations for the stable Q-model category structure of Γ-spaces we use the sets I and J from the proof of Theorem 1·5 in Appendix A. What we need to know about I and J

is that all maps in these sets are injective and all source (and target) objects are countable Γ-spaces. So all the maps of topological Γ-spaces in $I_{\mathcal{GT}}$ and $J_{\mathcal{GT}}$ are objectwise inclusions of sub-CW-complexes. Thus all the maps of topological Γ-spaces in $I_{\mathcal{GT}}$-cof$_{\mathrm{reg}}$ and $J_{\mathcal{GT}}$-cof$_{\mathrm{reg}}$ are pointwise closed inclusions of relative CW-complexes. So the smallness hypotheses in Lemma B2 are satisfied.

It remains to construct a fibrant replacement functor Q for topological Γ-spaces. We let Q' be any fibrant replacement functor for the stable Q-model category structure of $\mathcal{GS}$ which has the property that the stable equivalence $Y \to Q'Y$ is injective. For a topological Γ-space X we define QX and the natural map $X \to QX$ by the pushout in $\mathcal{GT}$

$$\begin{array}{ccc} |\operatorname{Sing} X| & \longrightarrow & |Q'\left(\operatorname{Sing} X\right)| \\ \downarrow & & \downarrow \\ X & \longrightarrow & QX \end{array}$$

The left vertical map is an objectwise weak equivalence and the upper horizontal map is objectwise an inclusion of a sub-CW-complex. Hence the right vertical map is also an objectwise weak equivalence and QX is in fact very special, thus stably fibrant. Since the upper horizontal map is a stable equivalence and both vertical maps are objectwise weak equivalences, the map $X \to QX$ is also a stable equivalence. So Lemma B2 applies and shows that topological Γ-spaces indeed form a stable model category. It then follows that the geometric realization and singular complex functors induce a Quillen equivalence.

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