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Homotopy Invariance of Convolution Products

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The purpose of this paper is to show that various convolution products are fully homotopical, meaning that they preserve weak equivalences in both variables without any cofibrancy hypothesis. We establish this property for diagrams of simplicial sets indexed by the category of finite sets and injections and for tame M-simplicial sets, with M the monoid of injective self-maps of the positive natural numbers. We also show that a certain convolution product studied by Nikolaus and the 1st author is fully homotopical. This implies that every presentably symmetric monoidal ∞-category can be represented by a symmetric monoidal model category with a fully homotopical monoidal product.

1 Introduction

The convolution product of functors between symmetric monoidal categories was introduced by the category theorist Brian Day [2]. It made a prominent appearance in homotopy theory when Jeff Smith and Manos Lydakis simultaneously and independently introduced the smash products of symmetric spectra [8] and the smash product of Γ-spaces [15], respectively. Since then, convolution products have become an essential ingredient in the homotopy theory toolkit and many more examples were introduced and studied in the context of stable homotopy theory [14, 17], unstable homotopy theory [11, 26, 27], equivariant homotopy theory [3, 16], motivic homotopy theory [4, 9], and ∞-category theory [7], to name just a few.

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In order for convolution products to be homotopically meaningful, they must "interact nicely" with some relevant notion of weak equivalence (except in the ∞-categorical case, where this is an intrinsic feature). In typical situations, homotopically useful convolution products come with compatible closed model structures. The compatibility then includes the "pushout product property", which implies that the convolution product is homotopy invariant when all objects involved are cofibrant. This cofibrancy requirement for the homotopy invariance often leads to cofibrancy hypotheses in applications. The proof of the homotopy invariance for cofibrant objects typically proceeds by a cellular reduction argument to free or representable objects.

Given the history of the subject, one would not expect convolution products to be *fully homotopical*, that is, to preserve the relevant weak equivalences in both variables without any cofibrancy hypothesis. Nevertheless, there are already two non-obvious instances of full homotopy invariance. On the one hand, the box product of orthogonal spaces (also known as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces) is fully homotopical [23, Theorem 1.3.2]. On the other hand, the operadic product of $\mathcal{L}$-spaces, that is, spaces with a continuous action of the topological monoid $\mathcal{L}$ of linear self-isometries of $\mathbb{R}^\infty$, is fully homotopical [24, Theorem 1.21]. In both cases, the weak equivalences can be chosen to be the global equivalences, and the proofs make essential use of explicit homotopies that are not available for more discrete or combinatorial index categories.

The purpose of the present paper is to show that there are more interesting instances of fully homotopical convolution products.

$\mathcal{I}$-spaces

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{m} = \{1, \ldots, m\}$ and with morphisms the injections. An $\mathcal{I}$-*space* is a covariant functor from $\mathcal{I}$ to simplicial sets. We say that a map of $\mathcal{I}$-spaces $f\colon X \to Y$ is an $\mathcal{I}$-*equivalence* if the induced map of homotopy colimits $f_{h\mathcal{I}}\colon X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$ is a weak equivalence. The category of $\mathcal{I}$-spaces $\mathrm{sSet}^{\mathcal{I}}$ has a Day type convolution product $\boxtimes$, with $X \boxtimes Y$ defined as the left Kan extension of the object-wise cartesian product $(\mathbf{m}, \mathbf{n}) \mapsto X(\mathbf{m}) \times Y(\mathbf{n})$ along the concatenation $- \sqcup -\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$. The interest in $\mathcal{I}$-spaces comes from the fact that homotopy types of E_∞ spaces can be represented by commutative $\mathcal{I}$-space monoids, that is, by commutative monoids with respect to $\boxtimes$, and that many interesting E_∞ spaces arise from explicit commutative $\mathcal{I}$-space monoids (see, e.g., [26, §1.1]).

Our 1st main result states that $\boxtimes$ is fully homotopical.

Theorem 1.1. Let X be an $\mathcal{I}$-space. Then the functor $X \boxtimes -$ preserves $\mathcal{I}$-equivalences.

This can be viewed as a discrete analog of [23, Theorem 1.3.2]. The homotopy invariance of the $\boxtimes$-product of $\mathcal{I}$-spaces was previously established by Schlichtkrull and the 1st author under the additional hypothesis that one of the factors is flat cofibrant, see [26, Proposition 8.2]. With Theorem 1.1, several flatness hypotheses required in [27] turn out to be unnecessary. For example, [27, Propositions 2.20, 2.23, 2.27, 4.2, 4.8, Lemma 2.25, Corollaries 2.29 and 4.3] hold without the flatness hypotheses and [27, Theorem 1.2 and Lemma 4.12] hold without the cofibrancy hypothesis on the commutative $\mathcal{I}$-space monoid.

Tame M-spaces

Let M be the monoid of injective self-maps of the set of positive natural numbers. A *tame M-set* is a set with an M-action that satisfies a certain local finiteness condition (see Definition 2.2). Actions of M were studied by the 2nd author in [22] where it was shown that the homotopy groups of symmetric spectra admit tame M-actions that for example detect semistability.

A *tame M-space* is a simplicial set with an M-action that is tame in every simplicial degree. The resulting category $\mathrm{sSet}^M_{\mathrm{tame}}$ also has a convolution product $\boxtimes$ that is analogous to the operadic product of $\mathcal{L}$-spaces. We say that a map $f\colon X \to Y$ of tame M-spaces is an *M-equivalence* if it induces a weak equivalence $f_{hM}\colon X_{hM} \to Y_{hM}$ of homotopy colimits (which are given by bar constructions).

Our 2nd main result states that $\boxtimes$ is fully homotopical.

Theorem 1.2. Let X be a tame M-space. Then the functor $X \boxtimes -$ preserves M-equivalences between tame M-spaces.

This theorem can be viewed as a discrete analog of [24, Theorem 1.21] and we use it to prove Theorem 1.1. It is also interesting in itself: we show that the category of tame M-spaces is equivalent to the reflective subcategory of the category of $\mathcal{I}$-spaces given by the flat $\mathcal{I}$-spaces. This implies that tame M-spaces and M-equivalences provide a model for the homotopy theory of spaces, and that E_∞ spaces can be represented by strictly commutative monoids in $(\mathrm{sSet}^M_{\mathrm{tame}}, \boxtimes)$. We also show that commutative $\boxtimes$-monoids of tame M-spaces can be described as algebras over an injection operad, see Theorem A.13.

Presentably symmetric monoidal ∞-categories

We now leave the framework of specific multiplicative models for the homotopy theory of spaces studied so far and consider general homotopy theories with a symmetric monoidal product. One convenient way to encode these are the *symmetric monoidal ∞-categories* introduced by Lurie [13]. More precisely, we shall consider *presentably symmetric monoidal ∞-categories*. By definition, these are presentable ∞-categories with a symmetric monoidal structure that preserves colimits separately in each variable.

When passing to underlying ∞-categories, combinatorial symmetric monoidal model categories give rise to presentably symmetric monoidal ∞-categories [13, Example 4.1.3.6 and Proposition 4.1.3.10]. Nikolaus and the 1st author showed in [18] that conversely, every presentably symmetric monoidal ∞-category $\mathcal{C}$ is represented by a combinatorial symmetric monoidal model category $\mathcal{M}$. By construction, $\mathcal{M}$ is a left Bousfield localization of a certain *contravariant $\mathcal{I}$-model structure* on an over category $\mathrm{sSet}^{\mathcal{I}}/N$ with N a commutative $\mathcal{I}$-simplicial set. Here both N and the localization depend on $\mathcal{C}$. The symmetric monoidal product of $\mathcal{M}$ is induced by the $\boxtimes$-product on $\mathrm{sSet}^{\mathcal{I}}$ and the multiplication of N and can therefore also be viewed as a convolution product. Combining some of the results about the interaction of $\mathcal{I}$-spaces and tame M-spaces used to prove Theorems 1.1 and 1.2 with an analysis of the contravariant $\mathcal{I}$-model structure, we show that the convolution product on $\mathcal{M}$ is fully homotopical and thus provide the following stronger variant of [18, Theorem 1.1].

Theorem 1.3. Every presentably symmetric monoidal ∞-category is represented by a simplicial, combinatorial and left proper symmetric monoidal model category with a fully homotopical monoidal product.

The authors view this theorem as a surprising result since point-set level models for multiplicative homotopy theories tend to be not fully homotopical (or to be not known to have this property). It leads to symmetric monoidal model categories which one may not have expected to exist. For example, applying the theorem to the ∞-category of spectra provides a symmetric monoidal model for the stable homotopy category with a fully homotopical smash product. To the authors' knowledge, none of the common symmetric monoidal models for the stable homotopy category such as [6, 8, 14, 17] is known to have this property (see also the questions below). As another example, Theorem 1.3 implies that the homotopy theory underlying the derived category of a commutative ring k admits a fully homotopical model. The existence of non-trivial Tor-terms shows that in general, this fails badly for the usual tensor product of chain complexes of k-modules. As a last example, applying Theorem 1.3 to the presentably symmetric monoidal ∞-category of ∞-operads constructed by Lurie [13, Chapter 2.2.5] leads to a fully homotopical model for the category of topological operads with the Boardman–Vogt tensor product.

The model category provided by [18, Theorem 1.1] also has the desirable feature that it lifts to operad algebras over simplicial operads and that weak equivalences of operads induce Quillen equivalences between categories of operad algebras [18, Theorem 2.5]. This implies that like for positive model structures on diagram spectra or diagram spaces, E_∞ objects can be represented by strictly commutative ones. Together with the full homotopy invariance, this model thus provides a setup where homotopical algebra is particularly simple.

Open questions

The above discussion and the close connection between $\mathcal{I}$-spaces and symmetric spectra lead to the following question.

Question 1.4. Is the smash product of symmetric spectra of simplicial sets fully homotopical for the stable equivalences?

At the time of this writing, and to the best of the authors' knowledge, this question is open, and the authors would like to see this being sorted out. Analogously, one can consider symmetric spectra of topological spaces, or the orthogonal case:

Question 1.5. Is the smash product of orthogonal spectra fully homotopical for the stable equivalences?

Affirmative answers to these questions would make numerous cofibrancy assumptions in applications of symmetric or orthogonal spectra superfluous and thereby lead to substantial simplifications.

Organization

Section 2 provides combinatorial results about tame M-sets that are used in the rest of the paper. In Section 3 we study the homotopy theory of tame M-spaces and prove Theorem 1.2. Section 4 is about homotopical properties of the $\boxtimes$-product of $\mathcal{I}$-spaces and contains the proof of Theorem 1.1. In Section 5 we identify tame M-spaces with flat $\mathcal{I}$-spaces and construct model structures on these categories. Section 6 discusses contravariant model structures and provides the proof of Theorem 1.3. In Appendix A we identify commutative $\boxtimes$-monoids in tame M-sets with tame algebras over the injection operad, and we supply an alternative characterization of the $\boxtimes$-product of tame M-spaces as an operadic product.

Conventions

In this paper we follow a common abuse of language in that the term *$\mathcal{I}$-space* means a functor from the injection category $\mathcal{I}$ to the category of *simplicial sets* (as opposed to a functor to some category of topological spaces). Similarly, an *M-space* is a simplicial set with an action of the injection monoid M. The use of simplicial sets is essential for several of our arguments, and we can offer no new insight about the convolution products of $\mathcal{I}$-topological spaces and M-topological spaces.

2 The Structure of Tame M-sets

In this section we discuss tame M-sets, that is, sets equipped with an action of the injection monoid that satisfy a certain local finiteness condition. The main result is Theorem 2.11, which says that every tame M-set W decomposes as a disjoint union of M-subsets arising in a specific way from certain Σ_m-sets associated with W. The arguments of this section are combinatorial in nature, but they crucially enter into the homotopical analysis in the subsequent sections.

Definition 2.1. The *injection monoid M* is the monoid of injective self-maps of the set $\omega = \{1, 2, 3, \dots\}$ of positive natural numbers with monoid structure given by composition. An M-set is a set with a left M-action.

Definition 2.2. An element x of an M-set W is *supported* on a subset A of ω if the following condition holds: for every injection $f \in M$ that fixes A elementwise, the relation $fx = x$ holds. An M-set W is *tame* if every element is supported on some finite subset of ω.

We write Set^M for the category of M-sets and M-equivariant maps, and we denote by $\mathrm{Set}^M_{\mathrm{tame}}$ the full subcategory of tame M-sets.

Clearly, if x is supported on A and $A \subseteq B \subseteq \omega$, then x is supported on B. Every element is supported on all of ω. An element is supported on the empty set if and only if it is fixed by M.

Proposition 2.3. Let x be an element of an M-set W. If x is supported on two finite subsets A and B of ω, then x is supported on the intersection $A \cap B$.

Proof. We let $f \in M$ be an injection that fixes $A \cap B$ elementwise. We let m be the maximum of the finite set $A \cup B \cup f(A)$ and define $\sigma \in M$ as the involution that interchanges j with $j + m$ for all $j \in B - A$, that is,

$$\sigma(j) = \begin{cases} j + m & \text{for } j \in B - A, \\ j - m & \text{for } j \in (B - A) + m, \text{and} \\ j & \text{for } j \notin (B - A) \cup ((B - A) + m). \end{cases}$$

In particular, the map σ fixes the set A elementwise. Since A and $f(A)$ are both disjoint from $B + m$, we can choose a bijection $\gamma \in M$ such that

$$\gamma(j) = \begin{cases} f(j) & \text{if } j \in A, \text{and} \\ j & \text{for } j \in B + m. \end{cases}$$

Then f can be written as the composition

$$f = \sigma(\sigma\gamma\sigma)(\sigma\gamma^{-1}f).$$

In this decomposition the factors σ and $\sigma\gamma^{-1}f$ fix A pointwise, and the factor $\sigma\gamma\sigma$ fixes B pointwise. So

$$\sigma x = (\sigma\gamma\sigma)x = (\sigma\gamma^{-1}f)x = x$$

because x is supported on A and on B. This gives

$$fx = \sigma(\sigma\gamma\sigma)(\sigma\gamma^{-1}f)x = x.$$

Since f was any injection fixing $A \cap B$ elementwise, the element x is supported on $A \cap B$. ∎

Definition 2.4. Let x be an element of an M-set. The *support* of x is the intersection of all finite subsets of ω on which x is supported.

We write $\mathrm{supp}(x)$ for the support of x and agree that $\mathrm{supp}(x) = \omega$ if x is not finitely supported. Proposition 2.3 then shows that x is supported on its support $\mathrm{supp}(x)$. It is important that in Definition 2.4 the intersection is only over finite supporting subsets. Indeed every object is supported on the set $\omega - \{j\}$ for every $j \in \omega$

(because the only injection that fixes $\omega - \{j\}$ elementwise is the identity). So without the finiteness condition the intersection in Definition 2.4 would always be empty.

Proposition 2.5. Let W be an M-set and $x \in W$.

(i) If the injections $f, g \in M$ agree on $\mathrm{supp}(x)$, then $fx = gx$.

(ii) For every injection $f \in M$, the relation

$$\mathrm{supp}(fx) \;\subseteq\; f(\mathrm{supp}(x))$$

holds. If x is finitely supported, then $\mathrm{supp}(fx) = f(\mathrm{supp}(x))$, and fx is also finitely supported.

Proof. (i) If x is not finitely supported, then $\mathrm{supp}(x) = \omega$, so $f = g$ and there is nothing to show. If x is finitely supported, we can choose a bijection $h \in M$ that agrees with f and g on $\mathrm{supp}(x)$. Then $h^{-1}f$ and $h^{-1}g$ fix the support of x elementwise and hence $(h^{-1}f)x = x = (h^{-1}g)x$. Thus,

$$fx \;=\; h((h^{-1}f)x) \;=\; hx \;=\; h((h^{-1}g)x) \;=\; gx.$$

(ii) We let $g \in M$ be an injection that fixes $f(\mathrm{supp}(x))$ elementwise. Then gf agrees with f on $\mathrm{supp}(x)$, so

$$g(fx) \;=\; (gf)x \;=\; fx$$

by part (i). We have thus shown that fx is supported on $f(\mathrm{supp}(x))$.

If x is finitely supported, then $f(\mathrm{supp}(x))$ is finite, and this proves that fx is finitely supported and $\mathrm{supp}(fx) \subseteq f(\mathrm{supp}(x))$. For the reverse inclusion we choose $h \in M$ such that hf fixes $\mathrm{supp}(x)$ elementwise; then $(hf)x = x$. Applying the argument to h and fx (instead of f and x) gives

$$\mathrm{supp}(x) \;=\; \mathrm{supp}((hf)x) \;=\; \mathrm{supp}(h(fx)) \;\subseteq\; h(\mathrm{supp}(fx))$$

and thus

$$f(\mathrm{supp}(x)) \;\subseteq\; f(h(\mathrm{supp}(fx))) \;=\; (fh)(\mathrm{supp}(fx)) \;=\; \mathrm{supp}(fx).$$

The last equation uses that fh is the identity on the set $f(\mathrm{supp}(x))$, hence also the identity on the subset $\mathrm{supp}(fx)$. This proves the desired relation when x is finitely supported. ■

As for any monoid, the category of M-sets is complete and cocomplete, and limits and colimits are created on underlying sets. Proposition 2.5 (ii) shows that for every M-set W, the subset

$$W_\tau \;=\; \{x \in W \mid x \text{ has finite support}\}$$

is closed under the M-action, and hence a tame M-subset of W. A morphism $u\colon V \to W$ of M-sets preserves supports in the sense of the containment relation

$$\mathrm{supp}(ux) \;\subseteq\; \mathrm{supp}(x)$$

for all $x \in V$. In particular, M-linear maps send finitely supported elements to finitely supported elements. So if the M-action on V is tame, then every M-linear map $f\colon V \to W$ has image in W_τ. This shows part (i) of the following lemma whose other parts are then formal consequences.

Lemma 2.6.

(i) The functor

$$(-)_\tau\colon \mathrm{Set}^M \;\to\; \mathrm{Set}^M_{\mathrm{tame}}, \quad W \;\longmapsto\; W_\tau$$

is right adjoint to the inclusion, with the inclusion $W_\tau \to W$ being the counit of the adjunction.

(ii) The category of tame M-sets is cocomplete, and the forgetful functor to sets preserves colimits.

(iii) The category of tame M-sets is complete, and limits can be calculated by applying the functor $(-)_\tau$ to limits in the category of M-sets.

Our next aim is to prove that every tame M-set is a disjoint union of M-sets of the form $\mathcal{I}_m \times_{\Sigma_m} A_m$ for varying $m \geq 0$ and Σ_m-sets A_m.

Proposition 2.7. Let W be a tame M-set and $f \in M$. Then the action of f is an injective map $W \to W$, and its image consists precisely of those elements that are supported on the set $f(\omega)$.

Proof. For injectivity we consider any $x, y \in W$ with $fx = fy$. Since f is injective and x and y are finitely supported, we can choose $h \in M$ such that hf is the identity on the support of x and the support of y. Then $x = hfx = hfy = y$.

It remains to identify the image of the action of f. For all $x \in W$ we have

$$\mathrm{supp}(fx) \ \subseteq \ f(\mathrm{supp}(x)) \ \subseteq \ f(\omega)$$

so fx is supported on $f(\omega)$. Now suppose that $z \in W$ is supported on $f(\omega)$. Then f restricts to a bijection from $f^{-1}(\mathrm{supp}(z))$ to $\mathrm{supp}(z)$. We choose a bijection $g \in M$ such that fg is the identity on $\mathrm{supp}(z)$. Then $fgz = z$, and hence z is in the image of the action of f. ■

Proposition 2.7 can fail for non-tame M-sets. An example is given by letting $f \in M$ act on the set $\{0, 1\}$ as the identity if the image of f has finite complement, and setting $f(0) = f(1) = 0$ if its image has infinite complement.

Definition 2.8. Given a set A, we write $\mathcal{I}_A$ for the M-set of injective maps from A to ω, where the monoid M acts by postcomposition.

Example 2.9. The M-set $\mathcal{I}_\emptyset$ has only one element, so the M-action is necessarily trivial. If A is finite and non-empty, then $\mathcal{I}_A$ is countably infinite and the M-action is non-trivial, but tame: the support of an injection $\alpha\colon A \to \omega$ is its image $\alpha(A)$.

If A is a finite subset of ω, then the M-set $\mathcal{I}_A$ represents the functor of taking elements with support in A: the inclusion $\iota_A\colon A \to \omega$ is supported on A, and for every M-set W, the evaluation map

$$\mathrm{Hom}_M(\mathcal{I}_A, W) \ \xrightarrow{\cong} \ \{x \in W\colon \mathrm{supp}(x) \subseteq A\}, \quad \varphi \ \longmapsto \ \varphi(\iota_A)$$

is bijective.

When $A = \mathbf{m} = \{1, \ldots, m\}$, we write $\mathcal{I}_m = \mathcal{I}_{\mathbf{m}}$. The M-set $\mathcal{I}_m$ comes with a commuting right action of the symmetric group Σ_m by precomposition. So we can—and will—view $\mathcal{I}_m$ as a Σ_m-object in the category of tame M-sets. If K is any left Σ_m-set, we can form the tame M-set $\mathcal{I}_m \times_{\Sigma_m} K$ by coequalizing the two Σ_m-actions on the product $\mathcal{I}_m \times K$. This construction yields a functor

$$\mathcal{I}_m \times_{\Sigma_m} -\colon \mathrm{Set}^{\Sigma_m} \ \to \ \mathrm{Set}^M_{\mathrm{tame}}.$$

Now we prove that every tame M-set is a disjoint union of M-sets that arise from the functors $\mathcal{I}_m \times_{\Sigma_m} -$ for varying $m \geq 0$. For an M-set W and $m \geq 0$, we write

$$s_m(W) \ = \ \{x \in W \ \ \mathrm{supp}(x) = \mathbf{m}\}$$

for the subset of elements whose support is the set $\mathbf{m} = \{1, \ldots, m\}$. This is a Σ_m-invariant subset of W by Proposition 2.5 (ii). For example, $s_0(W)$ is the set of M-fixed elements. We alert the reader that the Σ_m-sets $s_m(W)$ are *not* functors in W: a morphism $f\colon V \to W$ of M-sets may decrease the support, and hence it need not take $s_m(V)$ to $s_m(W)$.

We define a morphism of M-sets

$$\psi_m^W\colon \mathcal{I}_m \times_{\Sigma_m} s_m(W) \to W \qquad \text{by} \qquad [\alpha, x] \longmapsto \tilde{\alpha}x \tag{2.10}$$

where $\tilde{\alpha} \in M$ is any injection that extends $\alpha\colon \mathbf{m} \to \omega$. This assignment is well-defined by Proposition 2.5 (i).

Theorem 2.11. For every tame M-set W, the morphisms ψ_m^W assemble into an isomorphism of M-sets

$$\coprod_{m \geq 0} \mathcal{I}_m \times_{\Sigma_m} s_m(W) \xrightarrow{\cong} W.$$

Proof. We write

$$C_m(W) = \{x \in W \ |\text{supp}(x)| = m\}$$

for the subset of elements whose support has cardinality m. This is an M-invariant subset by Proposition 2.5 (ii). Since the M-action on W is tame, W is the disjoint union of the M-subsets $C_m(W)$ for $m \geq 0$. The image of ψ_m^W is contained in $C_m(W)$, so it remains to show that the morphism

$$\psi_m^W\colon \mathcal{I}_m \times_{\Sigma_m} s_m(W) \to C_m(W)$$

is an isomorphism for every $m \geq 0$.

For surjectivity we consider any element $x \in C_m(W)$. Since the support of x has cardinality m, we can choose a bijection $h \in M$ such that $h(\text{supp}(x)) = \{1, \ldots, m\}$. Then $\text{supp}(hx) = h(\text{supp}(x)) = \{1, \ldots, m\}$ by Proposition 2.5 (ii), so hx belongs to $s_m(W)$. Moreover,

$$\psi_m^W[h^{-1}|_{\{1,\ldots,m\}}, hx] = h^{-1}(hx) = x$$

so the map ψ_m^W is surjective.

For injectivity we consider $f, g \in \mathcal{I}_m$ and $x, y \in s_m(W)$ such that $\psi_m^W[f,x] = \psi_m^W[g,y]$. Then

$$\operatorname{supp}(\psi_m^W[f,x]) = f(\operatorname{supp}(x)) = f(\{1,\dots,m\})$$

and similarly $\operatorname{supp}(\psi_m^W[g,y]) = g(\{1,\dots,m\})$. So the class $\psi_m^W[f,x] = \psi_m^W[g,y]$ is supported on the intersection of $f(\{1,\dots,m\})$ and $g(\{1,\dots,m\})$. Since the support of $\psi_m^W[f,x]$ has cardinality m, we conclude that $f(\{1,\dots,m\}) = g(\{1,\dots,m\})$. There is thus a unique element $\sigma \in \Sigma_m$ such that $g = f\sigma$. We extend σ to an element $\tilde{\sigma} \in M$ by fixing all numbers greater than m.

We choose a bijection $h \in M$ that extends f; then $h\tilde{\sigma}$ extends $f\sigma = g$. Hence,

$$x = h^{-1}(hx) = h^{-1}\psi_m^W[f,x] = h^{-1}\psi_m^W[g,y] = h^{-1}h\tilde{\sigma}y = \tilde{\sigma}y.$$

So we conclude that

$$[f,x] = [f,\tilde{\sigma}y] = [f\sigma,y] = [g,y].$$

This proves that the map ψ_m^W is injective, hence bijective. ∎

We now define the box product of M-sets.

Definition 2.12. Let X and Y be M-sets. The *box product* $X \boxtimes Y$ is the subset of the product consisting of those pairs $(x,y) \in X \times Y$ such that $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$.

As we explain in Proposition A.17, the box product of M-sets is closely related to an *operadic product*.

Proposition 2.13. Let X and Y be M-sets.

(i) The box product $X \boxtimes Y$ is an M-invariant subset of $X \times Y$.
(ii) If X and Y are tame, then so is $X \boxtimes Y$.

Proof. (i) For all $(x,y) \in X \times Y$ and all $f \in M$ we have

$$\operatorname{supp}(fx) \cap \operatorname{supp}(fy) \subseteq f(\operatorname{supp}(x)) \cap f(\operatorname{supp}(y)) = f(\operatorname{supp}(x) \cap \operatorname{supp}(y)).$$

So the pair $f(x,y) = (fx, fy)$ belongs to $X \boxtimes Y$ whenever (x,y) does. Part (ii) follows from the relation $\operatorname{supp}_{X\times Y}(x,y) = \operatorname{supp}_X(x) \cup \operatorname{supp}_Y(y)$. ∎

The associativity, symmetry and unit isomorphisms of the cartesian product of M-sets clearly restrict to the box product; for example, the associativity isomorphism is given by

$$(X \boxtimes Y) \boxtimes Z \xrightarrow{\cong} X \boxtimes (Y \boxtimes Z), \quad ((x,y),z) \longmapsto (x,(y,z)).$$

Hence, they inherit the coherence conditions required for a symmetric monoidal product. We thus conclude the following:

Proposition 2.14. The box product is a symmetric monoidal structure on the category of M-sets with respect to the associativity, symmetry and unit isomorphisms inherited from the cartesian product. The box product restricts to a symmetric monoidal structure on the category of tame M-sets. Every one-element M-set is a unit object for the box product.

Example 2.15. The tame M-set $\mathcal{I}_m$ of injective maps from $\mathbf{m} = \{1,\ldots,m\}$ to ω was discussed in Example 2.9. We define a morphism of M-sets

$$\mathcal{I}_{m+n} \to \mathcal{I}_m \times \mathcal{I}_n, \quad f \longmapsto (fi^1, fi^2)$$

where $i^1 \colon \mathbf{m} \to \mathbf{m}+\mathbf{n}$ is the inclusion and $i^2 \colon \mathbf{n} \to \mathbf{m}+\mathbf{n}$ sends i to $m+i$. This morphism is injective and its image consists precisely of those pairs (α,β) such that

$$\mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \alpha(\mathbf{m}) \cap \beta(\mathbf{n}) = \emptyset.$$

So the map restricts to an isomorphism of M-sets

$$\rho \colon \mathcal{I}_{m+n} \xrightarrow{\cong} \mathcal{I}_m \boxtimes \mathcal{I}_n.$$

Lemma 2.16. Let X and Y be tame M-sets. There is a bijection of sets $X \boxtimes Y \to X \times Y$ that is natural for M-equivariant maps in Y.

Proof. We write $Y_{[A]}$ for the subset of elements of Y that are supported on $\omega - A$. By definition, the underlying set of $X \boxtimes Y$ is the disjoint union, over $x \in X$, of the sets

$$\{x\} \times Y_{[\mathrm{supp}(x)]}$$

and this decomposition is natural for M-equivariant morphisms in Y. We let $f \in M$ be any injection with $f(\omega) = \omega - \mathrm{supp}(x)$. Then the action of f is a bijection from Y onto the subset $Y_{[\mathrm{supp}(x)]}$, by Proposition 2.7. So the underlying set of $X \boxtimes Y$ bijects with the underlying set of $X \times Y$. ■

We warn the reader that the bijection between $X \boxtimes Y$ and $X \times Y$ constructed in the previous lemma is in general neither M-equivariant nor natural in X.

Corollary 2.17. If X is a tame M-set, then $X \boxtimes - \colon \mathrm{Set}^M_{\mathrm{tame}} \to \mathrm{Set}^M_{\mathrm{tame}}$ preserves colimits.

3 Homotopy Theory of M-spaces

In this section we introduce two notions of equivalence for M-spaces, the *weak equivalences* and the *M-equivalences*, and we show that the box product of M-spaces is fully homotopical for both kinds of equivalences.

Definition 3.1.

(i) An *M-space* is a simplicial set equipped with a left action of the injection monoid M. We write sSet^M for the category of M-spaces.

(ii) An M-space is *tame* if for every simplicial dimension k, the M-set of k-simplices is tame. We write $\mathrm{sSet}^M_{\mathrm{tame}}$ for the full subcategory of sSet^M whose objects are the tame M-spaces.

Definition 3.2. A morphism $f \colon X \to Y$ of M-spaces is

(i) a *weak equivalence* if it is a weak equivalence of underlying simplicial sets after forgetting the M-action, and

(ii) an *M-equivalence* if the induced morphism on homotopy colimits (bar constructions) $f_{hM} \colon X_{hM} \to Y_{hM}$ is a weak equivalence of simplicial sets.

The weak equivalences of M-spaces are analogous to the π_*-isomorphisms of symmetric spectra, or the $\mathcal{N}$-equivalences of $\mathcal{I}$-spaces. The M-equivalences are the more important class of equivalences since the homotopy theory represented by tame M-spaces relative to M-equivalences is the homotopy theory of spaces (see Corollary 5.11 below). Dropping the tameness assumption, a result by Dugger [5, Theorem 5.2] implies that the M-equivalences are the weak equivalence in a model structure on sSet^M that is Quillen equivalent to spaces. This uses that the classifying space of M is weakly contractible [22, Lemma 5.2].

Every weak equivalence of M-spaces induces a weak equivalence on homotopy M-orbits; hence weak equivalences are in particular M-equivalences. The converse is not true: for every injective map $\alpha\colon \mathbf{m} \to \mathbf{n}$, the induced restriction morphism

$$\alpha^*\colon \mathcal{I}_n \to \mathcal{I}_m, \quad f \longmapsto f \circ \alpha$$

is an M-equivalence because the M-homotopy orbits of source and target are weakly contractible, by the following Example 3.3. However, for $m \neq n$, the morphism is not a weak equivalence.

Example 3.3. The M-set $\mathcal{I}_m$ of injective maps $\mathbf{m} \to \omega$ was discussed in Example 2.9. We claim that the simplicial set $(\mathcal{I}_m)_{hM}$ is weakly contractible. To this end we observe that $(\mathcal{I}_m)_{hM}$ is the nerve of the translation category T_m whose object set is $\mathcal{I}_m$, and where morphisms from f to g are all $h \in M$ such that $hf = g$. Given any two $f, g \in \mathcal{I}_m$, we can choose a bijection $h \in M$ such that $hf = g$, and then h is an isomorphism from f to g in T_m. Since all objects of the category T_m are isomorphic, its nerve is weakly equivalent to the classifying space of the endomorphism monoid of any of its objects. All these endomorphism monoids are isomorphic to the injection monoid M, so we conclude that the simplicial set $(\mathcal{I}_m)_{hM} \cong N(T_m)$ is weakly equivalent to the classifying space of the injection monoid M. The classifying space of M is weakly contractible by [22, Lemma 5.2], hence so is the simplicial set $(\mathcal{I}_m)_{hM}$.

If X and Y are M-spaces, their box product $X \boxtimes Y$ is given by the levelwise box product of M-sets, introduced in Definition 2.12. The next results show that the box product with any tame M-space preserves weak equivalences and M-equivalences.

Theorem 3.4. For every tame M-space X, the functor $X \boxtimes -$ preserves weak equivalences between tame M-spaces.

Proof. We start with the special case where X is a tame M-*set* (as opposed to an M-space). For an M-space Y, Lemma 2.16 implies that $X \boxtimes Y$ is isomorphic to the underlying simplicial set of $X \times Y$ with the isomorphism being natural in Y. So $X \boxtimes -$ preserves weak equivalences between tame M-spaces because $X \times -$ does.

Now we treat the general case. The box product is formed dimensionwise in the simplicial direction, and realization (i.e., diagonal) of bisimplicial sets preserves weak equivalences. So the general case follows from the special case of M-sets. ∎

Since $X \boxtimes Y$ was defined as an M-invariant subspace of the product $X \times Y$, the two projections restrict to morphisms of M-spaces

$$p^1\colon X \boxtimes Y \to X \qquad \text{and} \qquad p^2\colon X \boxtimes Y \to Y.$$

Theorem 3.5. For all tame M-spaces X and Y, the morphism

$$(p^1_{hM}, p^2_{hM}) \colon (X \boxtimes Y)_{hM} \ \to \ X_{hM} \times Y_{hM}$$

is a weak equivalence of simplicial sets.

Proof. We start with the special case where $X = \mathcal{I}_m/H$ and $Y = \mathcal{I}_n/K$ for some $m, n \geq 0$, for some subgroup H of Σ_m, and some subgroup K of Σ_n. By Example 2.15, the box product $\mathcal{I}_m \boxtimes \mathcal{I}_n$ is isomorphic to $\mathcal{I}_{m+n}$. Hence, by Example 3.3, source and target of the $(H \times K)$-equivariant morphism

$$(p^1_{hM}, p^2_{hM}) \colon (\mathcal{I}_m \boxtimes \mathcal{I}_n)_{hM} \ \to \ (\mathcal{I}_m)_{hM} \times (\mathcal{I}_n)_{hM}$$

are weakly contractible. The actions of H on $(\mathcal{I}_m)_{hM}$ and of K on $(\mathcal{I}_n)_{hM}$ are free. Similarly, the isomorphism $(\mathcal{I}_m \boxtimes \mathcal{I}_n)_{hM} \cong (\mathcal{I}_{m+n})_{hM}$ provided by Example 2.15 shows that the action of $H \times K$ on $(\mathcal{I}_m \boxtimes \mathcal{I}_n)_{hM}$ is free. Any equivariant map between free $(H \times K)$-spaces that is a weak equivalence of underlying simplicial sets induces a weak equivalence on orbit spaces. So the morphism

$$(p^1_{hM}, p^2_{hM})/(H \times K) \colon (\mathcal{I}_m \boxtimes \mathcal{I}_n)_{hM}/(H \times K) \ \to \ (\mathcal{I}_m)_{hM}/H \times (\mathcal{I}_n)_{hM}/K$$

is a weak equivalence of simplicial sets. The claim now follows because the box product preserves colimits in each variable by Corollary 2.17, and because the canonical morphism

$$((\mathcal{I}_m)_{hM})/H \ \to \ (\mathcal{I}_m/H)_{hM}$$

is an isomorphism of simplicial sets.

Now we assume that X and Y are M-sets. Then X is isomorphic to a disjoint union of M-sets of the form $\mathcal{I}_m \times_{\Sigma_m} A_m$, for varying $m \geq 0$ and Σ_m-sets A_m, by Theorem 2.11. The Σ_m-set A, in turn, is isomorphic to a disjoint union of Σ_m-sets of the form Σ_m/H, for varying subgroups H of Σ_m. The box product of M-spaces and the cartesian product of simplicial sets distribute over disjoint unions, and homotopy orbits preserve disjoint unions. So source and target of the morphism under consideration decompose as disjoint unions of M-sets considered in the previous paragraph. Weak equivalences of simplicial sets are stable under disjoint unions, so this reduces the case of M-sets to the case of the previous paragraph.

Now we treat the general case. All relevant construction commute with realization of simplicial objects in M-spaces (i.e., diagonal of simplicial sets). So the general case follows from the special case of M-sets because realization of bisimplicial sets is homotopical. ■

This result implies Theorem 1.2 from the Introduction:

Proof of Theorem 1.2. Since the product of simplicial sets preserves weak equivalences in both variables, it follows from Theorem 3.5 that the $\boxtimes$-product is fully homotopical with respect to the M-equivalences. ■

Remark 3.6. The cartesian product of M-spaces, with diagonal M-action, is *not* homotopical for M-equivalences. For example, the unique morphism $\mathcal{I}_1 \to *$ to the terminal M-space is an M-equivalence by Example 3.3. However, the product $\mathcal{I}_1 \times \mathcal{I}_1$ is the disjoint union of $\mathcal{I}_1 \boxtimes \mathcal{I}_1$ (which is isomorphic to $\mathcal{I}_2$) and the diagonal (which is isomorphic to $\mathcal{I}_1$). So $(\mathcal{I}_1 \times \mathcal{I}_1)_{hM}$ consists of two weakly contractible components, and the projection to the 1st factor

$$\mathcal{I}_1 \times \mathcal{I}_1 \to \mathcal{I}_1$$

is *not* an M-equivalence.

We call a tame M-space X *semistable* if the canonical map $X \to X_{hM}$ is a weak equivalence of simplicial sets. This notion is analogous to semistability of symmetric spectra (see [8, §5.6] and [22, §4]) and $\mathcal{I}$-spaces (see [27, §2.5]). For semistable tame M-spaces X and Y, one can show that the inclusion $X \boxtimes Y \to X \times Y$ is both a weak equivalence and an M-equivalence. Hence, the product of semistable tame M-spaces is fully homotopical for M-equivalences.

Example 3.7 (Homotopy infinite symmetric product). We review an interesting class of tame M-spaces due to Schlichtkrull [21]. Schlichtkrull's paper is written for topological spaces, but we work with simplicial sets instead. We let X be a based simplicial set, with basepoint denoted by $*$. We define X^∞ as the simplicial subset of X^ω consisting, in each simplicial dimension, of those functions $\alpha\colon \omega \to X$ such that $\alpha(i) = *$ for almost all $i \in \omega$. The injection monoid M acts on X^∞ by

$$(f\alpha)(j) = \begin{cases} \alpha(i) & \text{if } f(i) = j, \text{and} \\ * & \text{if } j \notin f(\omega). \end{cases}$$

So informally speaking, one can think of $f\alpha$ as the composite $\alpha\circ f^{-1}$, with the caveat that f need not be invertible. An element $\alpha \in X^\infty$ is supported on the finite set $\omega - \alpha^{-1}(*)$, so the M-action on X^∞ is tame. Schlichtkrull's main result about this construction is that for connected X, the M-homotopy orbit space X^∞_{hM} is weakly equivalent to $Q(X)$, the underlying infinite loop space of the suspension spectrum of X, see [21, Theorem 1.2]. In other words, X^∞ is a model for $Q(X)$ in the world of tame M-spaces.

The M-space X^∞ has additional structure: it is a commutative monoid for the box product of M-spaces. Indeed, a multiplication

$$\mu_X\colon X^\infty \boxtimes X^\infty \ \to \ X^\infty$$

is given by

$$\mu_X(\alpha,\beta)(i) \ = \ \begin{cases} \alpha(i) & \text{if } \alpha(i)\neq *, \text{and} \\ \beta(i) & \text{if } \beta(i)\neq *, \text{and} \\ * & \text{if } \alpha(i)=\beta(i)=*. \end{cases}$$

This assignment makes sense because whenever α and β have disjoint support, then for all $i\in\omega$, at least one of the elements $\alpha(i)$ and $\beta(i)$ must be the basepoint of X. The multiplication map μ_X is clearly associative and commutative, and the constant map to the basepoint of X is a neutral element. With this additional structure, X^∞ has the following universal property. We let M act on $\omega_+\wedge X$ by $f(i,x) = (f(i),x)$. Then X^∞ is the free commutative $\boxtimes$-monoid, in the category of based tame M-spaces, generated by the tame M-space $\omega_+\wedge X$. This universal property, or direct inspection, shows that the functor

$$\mathrm{sSet}_* \ \to \ \mathrm{sSet}^M_{\mathrm{tame}}, \quad X \ \longmapsto \ X^\infty$$

from based simplicial sets takes coproducts to box products (which are coproducts in the category of commutative $\boxtimes$-monoids). More precisely, for all based simplicial sets X and Y, the composite map

$$X^\infty\boxtimes Y^\infty \xrightarrow{i_X^\infty\boxtimes i_Y^\infty} (X\vee Y)^\infty\boxtimes(X\vee Y)^\infty \xrightarrow{\mu_{X\vee Y}} (X\vee Y)^\infty$$

is an isomorphism of tame M-spaces, where $i_X\colon X\to X\vee Y$ and $i_Y\colon Y\to X\vee Y$ are the inclusions. The analogous statement for $\mathcal{I}$-spaces is [21, Lemma 3.1].

4 The Box Product of $\mathcal{I}$-spaces

In this section we prove that the box product of $\mathcal{I}$-spaces is fully homotopical for the classes of $\mathcal{I}$-equivalences and $\mathcal{N}$-equivalences. Our strategy is to reduce each of these claims to the corresponding one for the box product of tame M-spaces. The key tool is a certain functor from $\mathcal{I}$-spaces to tame M-spaces that matches the box products and the relevant notions of equivalences.

We let $\mathcal{I}$ denote the category whose objects are the sets $\mathbf{m} = \{1, \dots, m\}$ for $m \geq 0$, with $\mathbf{0}$ being the empty set. Morphisms in $\mathcal{I}$ are all injective maps. The category $\mathcal{I}$ is thus a skeleton of the category of finite sets and injections. (Other authors denote this category by $\mathbb{I}$ or *FI*, and some use the letter $\mathcal{I}$ for the orthogonal counterpart.)

Definition 4.1. An *$\mathcal{I}$-space* is a functor from the category $\mathcal{I}$ to the category of simplicial sets. A morphism of $\mathcal{I}$-spaces is a natural transformation of functors, and we write $\mathrm{sSet}^{\mathcal{I}}$ for the category of $\mathcal{I}$-spaces.

We denote by $\mathcal{N}$ the non-full subcategory of $\mathcal{I}$ containing all objects, but only the inclusions as morphisms. In other words, $\mathcal{N}$ is the category associated with the partially ordered set $(\mathbb{N}, \leq)$.

Construction 4.2 (From $\mathcal{I}$-spaces to tame M-spaces). We let X be an $\mathcal{I}$-space. We write

$$X(\omega) \;=\; \mathrm{colim}_{\mathcal{N}} X$$

for the colimit of X formed over the non-full subcategory $\mathcal{N}$. We observe that the simplicial set $X(\omega)$ supports a natural action by the injection monoid M, defined as follows. We let $[x] \in X(\omega)$ be represented by $x \in X(\mathbf{m})$, and we let $f \in M$ be an injection. We set $n = \max(f(\mathbf{m}))$ and write $\tilde{f} \colon \mathbf{m} \to \mathbf{n}$ for the restriction of f. Then we get a well-defined M-action by setting

$$f[x] \;=\; [X(\tilde{f})(x)].$$

Moreover, x is supported on $\mathbf{m}$, and so this M-action is tame.

In Construction 5.5 below we will exhibit this construction as a left adjoint functor $(-)(\omega) \colon \mathrm{Set}^{\mathcal{I}} \to \mathrm{Set}^{M}_{\mathrm{tame}}$. To analyze its homotopical properties, we use the following notions.

Definition 4.3. A morphism $f\colon X \to Y$ of $\mathcal{I}$-spaces is

- an $\mathcal{N}$-*isomorphism* if the induced map $f(\omega)\colon X(\omega) \to Y(\omega)$ on colimits over $\mathcal{N}$ is an isomorphism,
- an $\mathcal{N}$-*equivalence* if $f(\omega)$ is a weak equivalence of underlying simplicial sets, and
- an $\mathcal{I}$-*equivalence* if the induced map $f_{h\mathcal{I}}\colon X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$ on Bousfield-Kan homotopy colimits [1, Chapter XII] is a weak equivalence of simplicial sets.

Remark 4.4. The above definition of $\mathcal{N}$-equivalences is not the same as the one given in [27, § 2.5], where the homotopy colimit over $\mathcal{N}$ is used instead of the categorical colimit. However, sequential colimits of simplicial sets are fully homotopical for weak equivalences. (One way to see this is that sequential colimits of acyclic Kan fibrations are again acyclic Kan fibrations by a lifting argument, and that therefore cofibrant replacements in the projective model structure on $\mathrm{sSet}^{\mathcal{N}}$ are mapped to weak equivalences by $\mathrm{colim}_{\mathcal{N}}$; compare also Lemma 6.3.) This implies that for every functor $X\colon \mathcal{N} \to \mathrm{sSet}$, the canonical morphism

$$\mathrm{hocolim}_{\mathcal{N}} X \;\to\; \mathrm{colim}_{\mathcal{N}} X$$

is a weak equivalence, compare [1, Chapter XII, 3.5]. So the two definitions of $\mathcal{N}$-equivalences of $\mathcal{I}$-spaces are equivalent.

We recall that the homotopy colimit of an $\mathcal{I}$-space X is related to the homotopy colimit of the M-space $X(\omega)$ by a chain of three natural weak equivalences; this observation is due to Jeff Smith and recorded in [25, Proposition 2.2.9]. We let $\mathcal{I}_\omega$ denote the category whose objects are the sets $\mathbf{m} = \{1, \dots, m\}$ for $m \geq 0$, and the set $\omega = \{1, 2, 3, \dots\}$. Morphisms in $\mathcal{I}_\omega$ are all injective maps. The restriction functor

$$\mathrm{sSet}^{\mathcal{I}_\omega} \;\to\; \mathrm{sSet}^{\mathcal{I}}$$

has a left adjoint

$$\mathrm{sSet}^{\mathcal{I}} \;\to\; \mathrm{sSet}^{\mathcal{I}_\omega}, \quad X \longmapsto \tilde{X}$$

given by left Kan extension. Since $\mathcal{I}$ is a full subcategory of $\mathcal{I}_\omega$ and the canonical functor $\mathcal{N} \to \mathcal{I}/\omega$ is homotopy cofinal, we can take $\tilde{X} = X$ on the subcategory $\mathcal{I}$, and

$$\tilde{X}(\omega) \;=\; X(\omega) \;=\; \mathrm{colim}_{\mathcal{N}} X.$$

The inclusion $j\colon M \to \mathcal{I}_\omega$ of categories is homotopy cofinal. Writing L_hX for the homotopy left Kan extension of X along $\mathcal{I} \to \mathcal{I}_\omega$, this implies that the canonical morphism

$$((L_hX)(\omega))_{hM} = \mathrm{hocolim}_M(L_hX \circ j) \to \mathrm{hocolim}_{\mathcal{I}_\omega} L_hX$$

is another weak equivalence of simplicial sets, see the dual version of [1, Chapter XI, 9.2]. Because sequential colimits of simplicial sets are fully homotopical, the natural morphism from the homotopy Kan extension of X at ω to the (categorical) Kan extension at ω is a weak equivalence. So we obtain a chain of natural weak equivalences

$$X_{h\mathcal{I}} \xleftarrow{\sim} (L_hX)_{h\mathcal{I}_\omega} \xleftarrow{\sim} ((L_hX)(\omega))_{hM} \xrightarrow{\sim} X(\omega)_{hM}. \tag{4.5}$$

This chain of equivalences proves the 2nd item of the following proposition. The 1st item is a restatement of definitions.

Proposition 4.6.

(i) A morphism $f\colon X \to Y$ of $\mathcal{I}$-spaces is an $\mathcal{N}$-equivalence if and only if the morphism $f(\omega)\colon X(\omega) \to Y(\omega)$ is a weak equivalence of M-spaces.

(ii) A morphism $f\colon X \to Y$ of $\mathcal{I}$-spaces is an $\mathcal{I}$-equivalence if and only if the morphism $f(\omega)\colon X(\omega) \to Y(\omega)$ is an M-equivalence of M-spaces.

The category $\mathcal{I}$ supports a permutative structure

$$- \sqcup -\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$$

by concatenation of finite sets. On objects, it is given by $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$. The concatenation of two morphisms $\alpha\colon \mathbf{m} \to \mathbf{k}$ and $\beta\colon \mathbf{n} \to \mathbf{l}$ is given by

$$(\alpha \sqcup \beta)(i) = \begin{cases} \alpha(i) & \text{for } 1 \le i \le m, \text{and} \\ \beta(i-m)+k & \text{for } m+1 \le i \le m+n. \end{cases}$$

The *box product* $X \boxtimes Y$ of two $\mathcal{I}$-spaces X and Y is the convolution product for the concatenation monoidal structure, that is, the left Kan extension of the functor

$$\mathcal{I} \times \mathcal{I} \to \mathrm{sSet}, \quad (\mathbf{m}, \mathbf{n}) \longmapsto X(\mathbf{m}) \times Y(\mathbf{n})$$

along $- \sqcup -\colon \mathcal{I} \times \mathcal{I} \to \mathcal{I}$.

The box product of $\mathcal{I}$-spaces comes with two “projections”, that is, morphisms of $\mathcal{I}$-spaces

$$q^1 \colon X \boxtimes Y \ \to\ X \qquad \text{and} \qquad q^2 \colon X \boxtimes Y \ \to\ Y.$$

The morphism q^1 corresponds, via the universal property of the box product, to the bimorphism given by the composite

$$X(\mathbf{m}) \times Y(\mathbf{n}) \xrightarrow{\text{project}} X(\mathbf{m}) \xrightarrow{X(i^1)} X(\mathbf{m} \sqcup \mathbf{n}),$$

for $m, n \geq 0$, where $i^1 \colon \mathbf{m} \to \mathbf{m} \sqcup \mathbf{n}$ is the inclusion. The morphism q^2 corresponds, via the universal property of the box product, to the bimorphism given by the composite

$$X(\mathbf{m}) \times Y(\mathbf{n}) \xrightarrow{\text{project}} Y(\mathbf{n}) \xrightarrow{Y(i^2)} Y(\mathbf{m} \sqcup \mathbf{n}),$$

where $i^2 \colon \mathbf{n} \to \mathbf{m} \sqcup \mathbf{n}$ sends i to $m + i$.

We introduced the box product of M-spaces in Definition 2.12 as the M-invariant subspace of disjointly supported pairs inside the product. The image of the morphism of M-spaces

$$(q^1(\omega), q^2(\omega)) \colon (X \boxtimes Y)(\omega) \ \to\ X(\omega) \times Y(\omega)$$

lands in the subspace $X(\omega) \boxtimes Y(\omega)$, and we write

$$\tau_{X,Y} \colon (X \boxtimes Y)(\omega) \ \to\ X(\omega) \boxtimes Y(\omega)$$

for the restriction of $(q^1(\omega), q^2(\omega))$ to this image. The next proposition says that the transformation τ is a strong symmetric monoidal structure on the functor

$$(-)(\omega) \colon \mathrm{sSet}^{\mathcal{I}} \ \to\ \mathrm{sSet}^{M}_{\mathrm{tame}}.$$

Proposition 4.7. For all $\mathcal{I}$-spaces X and Y, the morphism $\tau_{X,Y} \colon (X \boxtimes Y)(\omega) \to X(\omega) \boxtimes Y(\omega)$ is an isomorphism of M-spaces.

Proof. All structure is sight is defined dimensionwise in the simplicial direction, so it suffices to prove the claim for $\mathcal{I}$-sets (as opposed to $\mathcal{I}$-spaces). Both sides of the transformation τ preserve colimits in each variable, which reduces the claim to representable $\mathcal{I}$-sets. The convolution product of represented functors is represented, that is,

$$\mathcal{I}(\mathbf{m},-) \boxtimes \mathcal{I}(\mathbf{n},-) \cong \mathcal{I}(\mathbf{m} \sqcup \mathbf{n},-),$$

and $\mathcal{I}(\mathbf{m},-)(\omega)$ is isomorphic to the M-set $\mathcal{I}_m$ discussed in Example 2.9. So the case of represented functors is taken care of by Example 2.15. ■

The next statement in particular contains Theorem 1.1 from the introduction.

Theorem 4.8. For every $\mathcal{I}$-space X, the functor $X \boxtimes -$ preserves $\mathcal{N}$-equivalences and $\mathcal{I}$-equivalences of $\mathcal{I}$-spaces.

Proof. We let $f\colon Y \to Z$ be a morphism of $\mathcal{I}$-spaces that is an $\mathcal{N}$-equivalence or $\mathcal{I}$-equivalence, respectively. Then by Proposition 4.6, the morphism of tame M-spaces $f(\omega)\colon Y(\omega) \to Z(\omega)$ is a weak equivalence or M-equivalence, respectively. So the morphism

$$X(\omega) \boxtimes f(\omega)\colon X(\omega) \boxtimes Y(\omega) \to X(\omega) \boxtimes Z(\omega)$$

is a weak equivalence by Theorem 3.4, or an M-equivalence by Theorem 1.2, respectively. By the isomorphism of Proposition 4.7, this means that

$$(X \boxtimes f)(\omega)\colon (X \boxtimes Y)(\omega) \to (X \boxtimes Z)(\omega)$$

is a weak equivalence or an M-equivalence, respectively. Another application of Proposition 4.6 proves the claim. ■

The next statement generalizes [27, Corollary 2.29].

Corollary 4.9. The morphism $(q^1_{h\mathcal{I}}, q^2_{h\mathcal{I}})\colon (X \boxtimes Y)_{h\mathcal{I}} \to X_{h\mathcal{I}} \times Y_{h\mathcal{I}}$ is a weak equivalence of simplicial sets for all $\mathcal{I}$-spaces X and Y.

Proof. Using the previous theorem, this follows from [27, Corollary 2.29] by arguing with a flat cofibrant replacement of X. Alternatively, one can use the chain of weak equivalences (4.5) to reduce this to a statement about tame M-spaces and then argue with Proposition 4.7 and Theorem 3.5 to directly verify the claim. ■

5 Flat $\mathcal{I}$-spaces and Tame M-spaces

In this section we show that the functor $\mathrm{colim}_{\mathcal{N}} = (-)(\omega)\colon \mathrm{sSet}^{\mathcal{I}} \to \mathrm{sSet}^{M}$ from Construction 4.2 is a left adjoint, and that it identifies a certain full subcategory of *flat* $\mathcal{I}$-sets with the category of tame M-sets. We will then use this adjunction for the homotopical analysis of flat $\mathcal{I}$-spaces and tame M-spaces.

Flat $\mathcal{I}$-sets

The following definition singles out a particular class of 'flat' $\mathcal{I}$-sets. The $\mathcal{I}$-spaces previously called "flat" are precisely the ones that are dimensionwise flat as $\mathcal{I}$-sets, see Remark 5.2.

Definition 5.1. An $\mathcal{I}$-set X is *flat* if the following two conditions hold:

(i) For every morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$, the map $X(\alpha)\colon X(\mathbf{m}) \to X(\mathbf{n})$ is injective.

(ii) The functor X sends pullback squares in $\mathcal{I}$ to pullback squares of sets.

Remark 5.2. The terminology just introduced is consistent with the usage of the adjective "flat" in the context of $\mathcal{I}$-spaces—despite the fact that the definitions look very different at 1st sight. We recall from [26, Definition 3.9] that an $\mathcal{I}$-space X is *flat* if for every $\mathbf{n}$ in $\mathcal{I}$, the latching morphism

$$\nu_n^X\colon L_{\mathbf{n}}X \;=\; \left(\mathrm{colim}_{\mathbf{m}\to\mathbf{n}\in\partial(\mathcal{I}/\mathbf{n})} X(\mathbf{m})\right) \;\to\; X(\mathbf{n}) \tag{5.3}$$

is a monomorphism of simplicial sets; here $\partial(\mathcal{I}/\mathbf{n})$ denotes the full subcategory of the over-category $(\mathcal{I}/\mathbf{n})$ on the objects that are non-isomorphisms. The flatness criterion established in [26, Proposition 3.11] precisely says that an $\mathcal{I}$-space is flat if and only if the $\mathcal{I}$-set of q-simplices is flat in the sense of Definition 5.1 for every $q \geq 0$.

The flat $\mathcal{I}$-spaces are relevant for us because they are the cofibrant objects of a *flat $\mathcal{I}$-model structure* on $\mathrm{sSet}^{\mathcal{I}}$ with weak equivalences the $\mathcal{I}$-equivalences [26, Proposition 3.10].

The following combinatorial property amounts to the fact that every monomorphism between flat $\mathcal{I}$-spaces is automatically a flat cofibration in the sense of [26, Definition 3.9], a fact that does seem to have been noticed before.

Proposition 5.4. Let $i\colon X \to Y$ be a monomorphism between flat $\mathcal{I}$-sets. Then for every $n \geq 0$, the induced map

$$\nu_n^Y \cup i(\mathbf{n})\colon L_n Y \cup_{L_n X} X(\mathbf{n}) \;\to\; Y(\mathbf{n})$$

is injective.

Proof. The argument in the proof of [26, Proposition 3.11] shows that the latching maps ν_n^X and ν_n^Y are injections. We claim that for every $n \geq 0$, the commutative square of sets

$$\begin{tikzcd} L_n X \arrow[r, "\nu_n^X"] \arrow[d, "L_n i"'] & X(\mathbf{n}) \arrow[d, "i(\mathbf{n})"] \\ L_n Y \arrow[r, "\nu_n^Y"'] & Y(\mathbf{n}) \end{tikzcd}$$

is a pullback. Since $i(\mathbf{n})$ is also injective, this implies that the pushout of the punctured square injects into $Y(\mathbf{n})$. Elements in the latching set $L_n Y$ are equivalence classes of pairs (α, y), where $\alpha\colon \mathbf{n-1} \to \mathbf{n}$ is an injection and $y \in Y(\mathbf{n-1})$. For the pullback property we consider an element $x \in X(\mathbf{n})$ such that

$$\nu_n^Y[\alpha, y] \;=\; \alpha_*(y) \;=\; i(\mathbf{n})(x)$$

holds in $Y(\mathbf{n})$. We let $\beta, \beta'\colon \mathbf{n} \to \mathbf{n+1}$ be the two injections that satisfy $\beta \circ \alpha = \beta' \circ \alpha$ and that differ on the unique element that is not in the image of α. Then

$$\begin{aligned} i(\mathbf{n+1})(\beta_*(x)) \;&=\; \beta_*(i(\mathbf{n})(x)) \;=\; \beta_*(\alpha_*(y)) \\ &=\; \beta'_*(\alpha_*(y)) \;=\; \beta'_*(i(\mathbf{n})(x)) \;=\; i(\mathbf{n+1})(\beta'_*(x)). \end{aligned}$$

Because i is a monomorphism, we conclude that $\beta_*(x) = \beta'_*(x)$. The following square is a pullback because X is flat:

$$\begin{tikzcd} X(\mathbf{n-1}) \arrow[r, "\alpha_*"] \arrow[d, "\alpha_*"'] & X(\mathbf{n}) \arrow[d, "\beta'_*"] \\ X(\mathbf{n}) \arrow[r, "\beta_*"'] & X(\mathbf{n+1}) \end{tikzcd}$$

So there is an element $z \in X(\mathbf{n}-\mathbf{1})$ such that $\alpha_*(z) = x$; equivalently, x is the image of $[\alpha, z]$ under the latching map $\nu_n^X \colon L_n X \to X(\mathbf{n})$. Now we also get

$$\nu_n^Y((L_n i)[\alpha, z]) \;=\; i(\mathbf{n})(\nu_n^X[\alpha, z]) \;=\; i(\mathbf{n})(x) \;=\; \nu_n^Y[\alpha, y].$$

Since Y is flat, its latching map ν_n^Y is injective, and we conclude that $(L_n i)[\alpha, z] = [\alpha, y]$ in $L_n Y$. This completes the proof of the pullback property, and hence the proof of the proposition. ■

Construction 5.5 (From M-sets to $\mathcal{I}$-sets). We let W be an M-set and $m \geq 0$. We write

$$W_{\mathbf{m}} \;=\; \{x \in W \mid \mathrm{supp}(x) \subseteq \mathbf{m}\}$$

for the set of elements that are supported on $\mathbf{m} = \{1, \dots, m\}$. Given an injection $\alpha : \mathbf{m} \to \mathbf{n}$, we choose an extension $\tilde{\alpha} \in M$, that is, such that $\tilde{\alpha}(i) = \alpha(i)$ for $1 \leq i \leq m$. We define

$$\alpha_* \colon W_{\mathbf{m}} \;\to\; W_{\mathbf{n}}$$

by $\alpha_*(x) = \tilde{\alpha}x$, and this is independent of the choice of extension by Proposition 2.5 (i). These assignments are functorial in α, that is, the entire data defines an $\mathcal{I}$-set $W_\bullet$. The inclusions $W_{\mathbf{m}} \subseteq W$ induce a natural morphism $\epsilon_W : (W_\bullet)(\omega) \to W$ of M-sets.

If X is an $\mathcal{I}$-set, we write $X^\flat = X(\omega)_\bullet$. The canonical map $X(\mathbf{m}) \to X(\omega)$ takes values in $X(\omega)_{\mathbf{m}}$; for varying m, these maps provide a natural transformation $\eta_X : X \to X(\omega)_\bullet = X^\flat$ of $\mathcal{I}$-sets.

Proposition 5.6.

(i) The morphisms $\eta_X \colon X \to X^\flat$ and $\epsilon_W \colon (W_\bullet)(\omega) \to W$ are the unit and counit of an adjunction

$$(-)(\omega) \colon \mathrm{Set}^{\mathcal{I}} \rightleftarrows \mathrm{Set}^M \colon (-)_\bullet \,.$$

(ii) The adjunction counit ϵ_W is injective, and ϵ_W is surjective if and only if W is tame.

(iii) The adjunction unit $\eta_X \colon X \to X^\flat$ is an $\mathcal{N}$-isomorphism.

(iv) The adjunction unit $\eta_X \colon X \to X^\flat$ is an isomorphism if and only if X is flat.

(v) The restrictions of $(-)(\omega)$ and $(-)_\bullet$ are an adjoint equivalence of categories

$$(-)(\omega)\colon \mathrm{Set}^{\mathcal{I}}_{\text{flat}} \rightleftarrows \mathrm{Set}^{M}_{\text{tame}} \colon (-)_\bullet$$

between the full subcategories of flat $\mathcal{I}$-sets and tame M-sets.

Proof. Parts (i) and (ii) are straightforward, and we omit their proofs.

(iii) The composite of $\eta_X(\omega)\colon X(\omega) \to (X(\omega)_\bullet)(\omega)$ with the adjunction counit $\epsilon_{X(\omega)}\colon (X(\omega)_\bullet)(\omega) \to X(\omega)$ is the identity. Since the M-action on $X(\omega)$ is tame, the counit $\epsilon_{X(\omega)}$ is an isomorphism by (ii). So the morphism $\eta_X(\omega)$ is an isomorphism.

(iv) We suppose first that η_X is an isomorphism. Then every morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces an injection $\alpha_*\colon W_{\mathbf{m}} \to W_{\mathbf{n}}$ by Proposition 2.7. Moreover, Proposition 2.3 shows that $W_\bullet$ preserves pullbacks. So the $\mathcal{I}$-set $W_\bullet$ is flat.

For the converse we suppose that X is flat. Then the maps $X(\iota_k^m)\colon X(\mathbf{k}) \to X(\mathbf{m})$ are injective, where $\iota_k^m\colon \mathbf{k} \to \mathbf{m}$ is the inclusion. So the canonical map $X(\mathbf{k}) \to \operatorname{colim}_{\mathcal{N}} X = X(\omega)$ is injective, hence so is its restriction $\eta_X(\mathbf{k})\colon X(\mathbf{k}) \to X(\omega)_{\mathbf{k}}$.

For surjectivity we consider any element of $X(\omega)_{\mathbf{k}}$, that is, an element of $X(\omega)$ that is supported on $\mathbf{k}$. We choose a representative $x \in X(\mathbf{m})$ of minimal dimension, that is, with $m \geq 0$ chosen as small as possible. We must show that $m \leq k$. We argue by contradiction and suppose that $m > k$. We let $d \in M$ be the injection defined by

$$d(i) = \begin{cases} i & \text{for } i < m, \text{and} \\ i+1 & \text{for } i \geq m. \end{cases}$$

Then $d[x] = [x]$ because d is the identity on $\mathbf{k}$ and $[x]$ is supported on $\mathbf{k}$. This means that the elements

$$X(\iota_m^{m+1})(x) \;\text{ and }\; X(d|_{\mathbf{m}})(x) \;\in\; X(\mathbf{m}+\mathbf{1})$$

represent the same element in the colimit $X(\omega)$. Since the canonical map from $X(\mathbf{m}+\mathbf{1})$ to $X(\omega)$ is injective, we conclude that $X(\iota_m^{m+1})(x) = X(d|_{\mathbf{m}})(x)$. Since X is flat, the following square is a pullback:

$$\begin{tikzcd}
X(\mathbf{m}-\mathbf{1}) \arrow[r, "X(\iota_{m-1}^m)"] \arrow[d, "X(\iota_{m-1}^m)"'] & X(\mathbf{m}) \arrow[d, "X(d|_{\mathbf{m}})"] \\
X(\mathbf{m}) \arrow[r, "X(\iota_m^{m+1})"'] & X(\mathbf{m}+\mathbf{1})
\end{tikzcd}$$

So there is an element $y \in X(\mathbf{m-1})$ such that $X(\iota^m_{m-1})(y) = x$. This contradicts the minimality of m, so we have shown that $m \le k$.

Part (v) is a formal consequence of the other statements: part (iii) implies that the restricted functor $(-)_\bullet \colon \mathrm{Set}^M_{\mathrm{tame}} \to \mathrm{Set}^{\mathcal{I}}$ is fully faithful, and part (iv) identifies its essential image as the flat $\mathcal{I}$-sets. ■

Homotopy theory of flat $\mathcal{I}$-spaces and tame M-spaces

In Proposition 5.6 we have exhibited flat $\mathcal{I}$-sets as a reflective subcategory inside all $\mathcal{I}$-sets, equivalent to the category of tame M-sets. Restricting to *tame* M-spaces and applying the relevant constructions in every simplicial degree provides an adjoint functor pair:

$$(-)(\omega)\colon \mathrm{sSet}^{\mathcal{I}} \rightleftarrows \mathrm{sSet}^M_{\mathrm{tame}} \colon (-)_\bullet$$

For an $\mathcal{I}$-space X, we write

$$X^\flat = (X(\omega))_\bullet$$

for the composite endofunctor of $\mathrm{sSet}^{\mathcal{I}}$. By Proposition 5.6, these constructions enjoy the following properties:

Corollary 5.7.

(i) The adjunction counit $\epsilon_W : (W_\bullet)(\omega) \to W$ is an isomorphism.

(ii) The adjunction unit $\eta_X \colon X \to X^\flat$ is an $\mathcal{N}$-isomorphism.

(iii) The adjunction unit $\eta_X \colon X \to X^\flat$ is an isomorphism if and only if X is flat.

(iv) The restrictions of $(-)(\omega)$ and $(-)_\bullet$ are equivalence of categories

$$(-)(\omega)\colon \mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}} \rightleftarrows \mathrm{sSet}^M_{\mathrm{tame}} \colon (-)_\bullet$$

between the full subcategories of flat $\mathcal{I}$-spaces and tame M-spaces.

Remark 5.8. Properties (ii) and (iii) in the corollary in particular say that $X^\flat$ is a “flat replacement” of X, in the sense that the adjunction unit $\eta_X \colon X \to X^\flat$ is an $\mathcal{N}$-isomorphism, and hence an $\mathcal{I}$-equivalence, with a flat target. This should be contrasted with cofibrant replacement in the flat model structure on $\mathcal{I}$-spaces, which provides an $\mathcal{I}$-equivalence (even a level equivalence) *from* a flat $\mathcal{I}$-space *to* X.

Remark 5.9. Corollary 5.7 (ii) and the strong monoidality of $(-)(\omega)$ established in Proposition 4.7 also lead to an alternative proof of Theorem 1.1: they imply that for every pair of $\mathcal{I}$-spaces X, Y, there is a natural $\mathcal{N}$-isomorphism $X \boxtimes Y \to X^\flat \boxtimes Y$ with $X^\flat$ a flat. The homotopy invariance of $\boxtimes$ then follows from [26, Proposition 8.2], which implies that $X^\flat \boxtimes -$ preserves $\mathcal{I}$-equivalences.

A relatively formal consequence of this setup is that the absolute and positive flat $\mathcal{I}$-model structures on $\mathrm{sSet}^{\mathcal{I}}$ [26, Proposition 3.10] restrict to model structures on the full subcategory $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ of flat $\mathcal{I}$-spaces. The *flat $\mathcal{I}$-fibrations* are defined in [26, Definition 3.9] as the morphisms of $\mathcal{I}$-spaces with the right lifting property against the class of flat $\mathcal{I}$-cofibrations that are also $\mathcal{I}$-equivalences. In [26, Section 6.11], the flat $\mathcal{I}$-fibrations are identified in more explicit terms.

Theorem 5.10.

(i) The classes of $\mathcal{I}$-equivalences, monomorphisms and flat $\mathcal{I}$-fibrations form the *flat $\mathcal{I}$-model structure* on the category $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ of flat $\mathcal{I}$-spaces.

(ii) The classes of $\mathcal{I}$-equivalences, monomorphisms that are also isomorphisms at $\mathbf{0}$, and positive flat $\mathcal{I}$-fibrations the *positive flat $\mathcal{I}$-model structure* on the category $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ of flat $\mathcal{I}$-spaces.

(iii) The flat and positive flat $\mathcal{I}$-model structures on $\mathrm{sSet}^{\mathcal{I}}$ are cofibrantly generated, proper and simplicial.

(iv) The inclusion $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}} \to \mathrm{sSet}^{\mathcal{I}}$ is the right Quillen functor in a Quillen equivalence between the flat $\mathcal{I}$-model structures.

Proof. (i) Being a reflective subcategory of $\mathrm{sSet}^{\mathcal{I}}$, the category $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ is complete with limits created by the inclusion and colimits created by applying $(-)^\flat$ to the corresponding colimits in $\mathrm{sSet}^{\mathcal{I}}$ (see, e.g., [20, Proposition 4.5.15]). Factorizations and the 2-out-of-3, retract and lifting properties are inherited from $\mathrm{sSet}^{\mathcal{I}}$. The same holds for the simplicial structure, properness and the generating (acyclic) cofibrations claimed in part (iii). Proposition 5.4 shows that every monomorphism between flat $\mathcal{I}$-spaces is already a flat cofibration. So the restriction of flat cofibrations to $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ indeed yields the class of monomorphisms. The proof of part (ii) is very similar to that of part (i), and we omit the details.

(iv) The Quillen equivalence statement is a direct consequence of the fact that the adjunction unit $X \to X^\flat$ is always an $\mathcal{N}$-isomorphism, and hence an $\mathcal{I}$-equivalence, by Proposition 5.6 (i). ∎

Analogously, the absolute and positive projective $\mathcal{I}$-model structures and the level model structures studied in [26, § 3] carry over to $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$.

Corollary 5.11.

(i) The classes of M-equivalences and monomorphisms are part of a cofibrantly generated, proper and simplicial model structure on the category $\mathrm{sSet}^{M}_{\mathrm{tame}}$ of tame M-spaces.

(ii) The classes of M-equivalences and monomorphisms that are also isomorphisms on M-fixed points are part of a cofibrantly generated, proper and simplicial model structure on the category $\mathrm{sSet}^{M}_{\mathrm{tame}}$ of tame M-spaces.

Proof. For parts (i) and (ii) we transport the two model structures of Theorem 5.10 along the equivalence of categories between flat $\mathcal{I}$-spaces and tame M-spaces provided by Proposition 5.6 (v). We note that Proposition 4.6 (ii) identifies the weak equivalences as the M-equivalences. ■

Remark 5.12. The results we proved about the homotopy theory of tame M-spaces imply that the two model structures of Corollary 5.11 are Quillen equivalent to the Kan–Quillen model structure on the category of simplicial sets. For easier reference, we spell out an explicit chain of two Quillen equivalences. Proposition 6.23 of [26] shows that the colimit functor $\mathrm{colim}_{\mathcal{I}} \colon \mathrm{sSet}^{\mathcal{I}} \to \mathrm{sSet}$ is a left Quillen equivalence for the projective model $\mathcal{I}$-model structure on the category of $\mathcal{I}$-spaces of [26, Propositions 3.2]. The projective and flat $\mathcal{I}$-model structures on $\mathrm{sSet}^{\mathcal{I}}$ have the same weak equivalences and nested classes of cofibrations, so they are Quillen equivalent. The flat $\mathcal{I}$-model structures on $\mathrm{sSet}^{\mathcal{I}}$ and on its full subcategory $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$ are Quillen equivalent by Theorem 5.10 (iv). And the model structure on $\mathrm{sSet}^{M}_{\mathrm{tame}}$ matches the model structure on $\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}}$, by design. If we combine all this, we arrive at a chain of two Quillen equivalences, with left adjoints depicted on top:

$$\mathrm{sSet} \underset{\mathrm{const}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\leftrightarrows}} \mathrm{sSet}^{\mathcal{I},\mathrm{proj}} \underset{(-)_{\bullet}}{\overset{(-)(\omega)}{\rightleftarrows}} \mathrm{sSet}^{M}_{\mathrm{tame}}.$$

The middle term has the projective $\mathcal{I}$-model structure, and simplicial sets carry the Kan–Quillen model structure.

The following theorem states that the positive model structure from Corollary 5.11 lifts to commutative monoids.

Theorem 5.13. The category $\mathit{Com}(\mathrm{sSet}^{M}_{\mathrm{tame}}, \boxtimes)$ of commutative $\boxtimes$-monoids in tame M-spaces admits a positive model structure with weak equivalences the M-equivalences. It is Quillen equivalent to the category of E_∞ spaces.

Since the functor $(-)(\omega)\colon \mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}} \to \mathrm{sSet}^{M}_{\mathrm{tame}}$ is a strong symmetric monoidal equivalence of categories, it induces an equivalence on the categories of commutative monoids. Thus, the theorem also gives a *positive flat $\mathcal{I}$-model structure* on the category of commutative monoids in $(\mathrm{sSet}^{\mathcal{I}}, \boxtimes)$ whose underlying $\mathcal{I}$-spaces are flat.

Proof of Theorem 5.13. The category $\mathrm{Com}(\mathrm{sSet}^{M}_{\mathrm{tame}}, \boxtimes)$ is complete because the underlying category $\mathrm{sSet}^{M}_{\mathrm{tame}}$ is, and limits in commutative monoids are created in the underlying category. Using [19, Proposition 2.3.5], cocompleteness follows because $\mathrm{sSet}^{M}_{\mathrm{tame}}$ is cocomplete by Lemma 2.6 (or the argument in the proof of Corollary 5.11) and $\boxtimes$ preserves colimits in each variable by Corollary 2.17.

The model structure now arises by restricting the flat $\mathcal{I}$-model structure on commutative $\mathcal{I}$-space monoids from [26, Proposition 3.15(i)] to the category of underlying flat commutative $\mathcal{I}$-space monoids and transporting it along the equivalence of categories to $\mathrm{Com}(\mathrm{sSet}^{M}_{\mathrm{tame}}, \boxtimes)$. The only non-obvious part is to get the factorizations. For this, it is sufficient to check that positive cofibrations in commutative $\mathcal{I}$-space monoids with underlying flat domain are absolute flat cofibrations of $\mathcal{I}$-spaces. This is a slightly stronger statement than [26, Proposition 12.5] and follows from the argument in the proof of [26, Lemma 12.17].

For the Quillen equivalence statement, we first note that since $(-)(\omega)$ is strong symmetric monoidal by Proposition 4.7, its right adjoint $(-)_{\bullet}$ is lax symmetric monoidal, and so the composite $(-)^{\flat}$ is also lax symmetric monoidal. Hence, $(-)^{\flat}$ also induces a left adjoint $\mathrm{Com}(\mathrm{sSet}^{\mathcal{I}}) \to \mathrm{Com}(\mathrm{sSet}^{\mathcal{I}}_{\mathrm{flat}})$ with right adjoint the inclusion. This adjunction is a Quillen equivalence with respect to the positive flat model structure since the underlying $\mathcal{I}$-spaces of cofibrant commutative $\mathcal{I}$-space monoids are flat. The claim follows since $\mathrm{Com}(\mathrm{sSet}^{\mathcal{I}})$ is Quillen equivalent to E_∞ spaces by [26, Theorem 3.6 and Proposition 9.8(ii)]. ■

6 Presentably Symmetric Monoidal ∞-categories

The aim of this section is to prove Theorem 1.3 from the introduction. The strategy of proof is to generalize the alternative approach to Theorem 1.1 outlined in Remark 5.9.

We let N be a commutative $\mathcal{I}$-space monoid. We write $\mathrm{sSet}^{\mathcal{I}}/N$ for the category of $\mathcal{I}$-spaces augmented over N. This over category $\mathrm{sSet}^{\mathcal{I}}/N$ inherits a symmetric monoidal convolution product given by

$$(X \to N) \boxtimes (Y \to N) = (X \boxtimes Y \to N \boxtimes N \to N)$$

where the last map is the multiplication of N. Since $(-)(\omega)\colon \mathrm{sSet}^{\mathcal{I}} \to \mathrm{sSet}^{M}_{\mathrm{tame}}$ is strong symmetric monoidal by Proposition 4.7, it induces a strong symmetric monoidal functor $\mathrm{sSet}^{\mathcal{I}}/N \to \mathrm{sSet}^{M}_{\mathrm{tame}}/N(\omega)$.

Corollary 6.1. For every object $X \to N$ in $\mathrm{sSet}^{\mathcal{I}}/N$, the endofunctor $(X \to N) \boxtimes -$ of $\mathrm{sSet}^{\mathcal{I}}/N$ preserves $\mathcal{N}$-isomorphisms.

Proof. The map obtained by applying $(-)(\omega)$ to the product of $X \to N$ with an $\mathcal{N}$-isomorphism $Y \to Y'$ over N is isomorphic to $X(\omega) \boxtimes (Y(\omega) \to Y'(\omega))$. ■

Contravariant model structures

If S is a simplicial set, the over category sSet/S admits a *contravariant model structure* [12, § 2.1.4]. It is characterized by the property that its cofibrations are the monomorphisms and its fibrant objects are the morphisms $K \to S$ with the right lifting property against $\{\Lambda^n_i \subseteq \Delta^n \mid 0 < i \leq n\}$.

We shall now consider $\mathcal{I}$-diagrams in sSet/S, and again call a morphism in $(\mathrm{sSet}/S)^{\mathcal{I}}$ an $\mathcal{N}$-isomorphism if it induces an isomorphism when passing to the colimit of the underlying $\mathcal{N}$-diagram. Moreover, we say that a map is a *contravariant $\mathcal{I}$-equivalence* if the homotopy colimit over $\mathcal{I}$ formed with respect the contravariant model structure sends it to a contravariant weak equivalence in sSet/S. Since the covariant model structure is simplicial by the dual of [12, Proposition 2.1.4.8], one may model this homotopy colimit by implementing the Bousfield–Kan formula.

Lemma 6.2. The $\mathcal{N}$-isomorphisms in $(\mathrm{sSet}/S)^{\mathcal{I}}$ are contravariant $\mathcal{I}$-equivalences.

Proof. Implementing the 1st two equivalences in (4.5) for the contravariant model structure shows that $\mathrm{hocolim}_{\mathcal{N}}$-equivalences are $\mathcal{I}$-equivalences. Thus, it is sufficient to verify that the canonical map $\mathrm{hocolim}_{\mathcal{N}} X \to \mathrm{colim}_{\mathcal{N}} X$ is a contravariant weak equivalence. For this it is in turn sufficient to show that if $f\colon X \to Y$ is a map of $\mathcal{N}$-diagrams in sSet/S with each $f(\mathbf{m})$ a contravariant weak equivalence, then $\mathrm{colim}_{\mathcal{N}} f$ is a contravariant weak equivalence. To see this, we factor f in the projective level model structure induced by the contravariant model structure as an acyclic cofibration h followed by an acyclic fibration g. Since $\mathrm{colim}_{\mathcal{N}}$ is left Quillen with respect to the projective level model structure, $\mathrm{colim}_{\mathcal{N}} h$ is a contravariant acyclic cofibration. Since the contravariant cofibrations coincide with the cofibrations of the over category model structure induced by the Kan model structure on sSet, the acyclic fibrations in the contravariant model structure are the maps that are acyclic Kan fibrations when forgetting the projection to S. Since acyclic Kan fibrations are characterized by having the right lifting property with respect to the set $\{\partial\Delta^n \subseteq \Delta^n \mid n \geq 0\}$ and $\partial\Delta^n$ and Δ^n are finite, it follows that $\mathrm{colim}_{\mathcal{N}} g$ is an acyclic Kan fibration when forgetting the projection to S. This shows that $\mathrm{colim}_{\mathcal{N}} f$ is a contravariant weak equivalence because it is the composite of a contravariant acyclic cofibration $\mathrm{colim}_{\mathcal{N}} h$ and an acyclic Kan fibration $\mathrm{colim}_{\mathcal{N}} g$. ■

Now we let Z be an $\mathcal{I}$-diagram of simplicial sets, and we consider the over category $\mathrm{sSet}^{\mathcal{I}}/Z$. This category admits a *positive contravariant $\mathcal{I}$-model structure* introduced in [18, Proposition 3.10]. It is defined as a left Bousfield localization of a positive contravariant level model structure, where maps $X \to Y$ in $\mathrm{sSet}^{\mathcal{I}}/Z$ are weak equivalences or fibrations if the maps $X(\mathbf{n}) \to Y(\mathbf{n})$ are weak equivalences or fibrations in the contravariant model structure on $\mathrm{sSet}/Z(\mathbf{n})$ for all $\mathbf{n}$ in $\mathcal{I}$ with $n \geq 1$. The reason for considering the positive contravariant $\mathcal{I}$-model structure is that it can be used to give a symmetric monoidal model for the contravariant model structure on simplicial sets over a symmetric monoidal ∞-category, see [18, Theorem 3.15].

The positive contravariant $\mathcal{I}$-model structure is somewhat difficult to work with since we are not aware of an intrinsic characterization of its weak equivalences. To identify the resulting homotopy category, we recall from [10, Theorem 3.1] that there is a *positive Joyal $\mathcal{I}$-model structure* on $\mathrm{sSet}^{\mathcal{I}}$ whose weak equivalences are the maps that induce weak equivalences in the Joyal model structure when forming the homotopy colimit with respect to the Joyal model structure. The key property of the positive contravariant $\mathcal{I}$-model structure is now that when Z is fibrant in the positive Joyal $\mathcal{I}$-model structure, then a zig-zag of Joyal $\mathcal{I}$-equivalences between positive fibrant objects relating Z to a constant $\mathcal{I}$-diagram on a simplicial set S induces a zig-zag of Quillen equivalences between $\mathrm{sSet}^{\mathcal{I}}/Z$ with the positive contravariant $\mathcal{I}$-model structure and sSet/S with the contravariant model structure [18, Lemma 3.11 and Corollary 3.14].

Lemma 6.3. If Z is an object of $\mathrm{sSet}^{\mathcal{I}}$ that is fibrant in the positive Joyal $\mathcal{I}$-model structure, then an $\mathcal{N}$-isomorphism in $\mathrm{sSet}^{\mathcal{I}}/Z$ is also a weak equivalence in the positive contravariant $\mathcal{I}$-model structure.

Proof. We let $Z^c \to Z$ be a cofibrant replacement of Z in the positive Joyal $\mathcal{I}$-model structure on $\mathrm{sSet}^{\mathcal{I}}$. Then we can assume $Z^c \to Z$ to be a positive level fibration so that $Z^c \to Z$ is an acyclic Kan fibration in positive degrees. Moreover, we write $S = \mathrm{colim}_{\mathcal{I}} Z^c$

and consider the adjunction unit $Z^c \to \mathrm{const}_{\mathcal{I}} S$. The latter map is a Joyal $\mathcal{I}$-equivalence by [10, Corollary 2.4] and thus a positive Joyal level equivalence since both Z^c and $\mathrm{const}_{\mathcal{I}} S$ are homotopy constant in positive degrees with respect to the Joyal model structure.

Given an object $X \to Z$ in $\mathrm{sSet}^{\mathcal{I}}/Z$, we get a composite of acyclic Kan fibrations

$$(X \times_{Z^c} Z)^c \xrightarrow{\sim} X \times_{Z^c} Z \xrightarrow{\sim} X,$$

where the 1st map is a positive $\mathcal{I}$-cofibrant replacement in $\mathrm{sSet}^{\mathcal{I}}/Z^c$ and the 2nd map is the base change of $Z^c \to Z$ along $X \to Z$. It follows that X is weakly equivalent to image of $(X \times_{Z^c} Z)^c$ under $(Z^c \to Z)_! \colon \mathrm{sSet}^{\mathcal{I}}/Z^c \to \mathrm{sSet}^{\mathcal{I}}/Z$. Since both $(Z^c \to Z)_!$ and $(Z^c \to \mathrm{const}_{\mathcal{I}} S)_!$ are Quillen equivalences [18, Proposition 3.10], we deduce that a map $f \colon X \to X'$ in $\mathrm{sSet}^{\mathcal{I}}/Z$ is a weak equivalence in the positive contravariant $\mathcal{I}$-model structure if and only if the image of $(f \times_{Z^c} Z)^c$ under $(Z^c \to \mathrm{const}_{\mathcal{I}} S)_!$ is.

Now assume that f is an $\mathcal{N}$-isomorphism. Since $\mathrm{colim}_{\mathcal{N}}$ commutes with pullbacks and sends maps that are acyclic Kan fibrations in positive levels to acyclic Kan fibrations, the image of $f \times_{Z^c} Z$ under $(Z^c \to \mathrm{const}_{\mathcal{I}} S)_!$ is an $\mathcal{N}$-isomorphism that is weakly equivalent in $\mathrm{sSet}^{\mathcal{I}}/\mathrm{const}_{\mathcal{I}} S$ to the image of $(f \times_{Z^c} Z)^c$ under $(Z^c \to \mathrm{const}_{\mathcal{I}} S)_!$. This reduces the claim to showing that an $\mathcal{N}$-isomorphism over a constant base is a positive contravariant $\mathcal{I}$-equivalence. In this case, the proof of [18, Proposition 3.13] shows that weak equivalences in the positive contravariant $\mathcal{I}$-model structure coincide with the $\mathrm{hocolim}_{\mathcal{I}}$-equivalences on $\mathcal{I}$-diagrams in sSet/S. Thus, the claim follows from Lemma 6.2. ■

Proposition 6.4. Let N be a commutative $\mathcal{I}$-simplicial set that is fibrant in the positive Joyal $\mathcal{I}$-model structure on $\mathrm{sSet}^{\mathcal{I}}$, and whose underlying $\mathcal{I}$-simplicial set is flat. Then the $\boxtimes$-product on $\mathrm{sSet}^{\mathcal{I}}/N$ is homotopy invariant with respect to the positive Joyal $\mathcal{I}$-model structure.

Proof. Because the flat $\mathcal{I}$-simplicial sets are reflective in $\mathcal{I}$-simplicial sets and the underlying $\mathcal{I}$-simplicial set of N is flat, every morphism $f \colon X \to N$ factors as $f = \bar{f} \circ \eta_X$ for a unique morphism $\bar{f} \colon X^\flat \to N$, where $\eta_X \colon X \to X^\flat$ is the adjunction unit. Then η_X is a morphism in $\mathrm{sSet}^{\mathcal{I}}/N$ from $f \colon X \to N$ to $\bar{f} \colon X^\flat \to N$.

Corollary 6.1 and Lemma 6.3 show that $\eta_X \colon (f \colon X \to N) \to (\bar{f} \colon X^\flat \to N)$ induces a weak equivalence $\eta_X \boxtimes (Y \to N)$ in the positive contravariant $\mathcal{I}$-model structure for every object $Y \to N$ in $\mathrm{sSet}^{\mathcal{I}}/N$. So it is sufficient to show that $(X \to N) \boxtimes -$ preserves weak equivalences in the positive contravariant $\mathcal{I}$-model structure when X is flat.

To see this, we first consider the case where X is of the form $X = K \times \mathcal{I}(\mathbf{k}, -)/H$, where K is a simplicial set, $\mathbf{k}$ is an object of $\mathcal{I}$, and $H \subseteq \Sigma_k$ is a subgroup. Since the Σ_k-action on

$$((K \times \mathcal{I}(\mathbf{k}, -)) \boxtimes Y)(\mathbf{m}) \;\cong\; K \times \operatorname{colim}_{\mathbf{k} \sqcup \mathbf{l} \to \mathbf{m}} X(\mathbf{l})$$

is levelwise free by [26, Lemma 5.7], it follows that $X \boxtimes -$ preserves absolute levelwise contravariant $\mathcal{I}$-equivalences. From here the proof proceeds as the one of [18, Proposition 3.18] that addresses the claim for X being absolute projective cofibrant (rather than flat). ■

Proof of Theorem 1.3. Proposition 6.4 implies that the monoidal product on the category $\mathrm{sSet}^{\mathcal{I}}/M^{\mathrm{rig}}$ considered in [18, Theorem 3.15] is homotopy invariant. Since the homotopy invariance allows us to cofibrantly replace objects, it is preserved under the monoidal left Bousfield localization arising from [18, Proposition 2.2]. Therefore, the proof of [18,Theorem 1.1] shows the claim. ■

Appendix A. Algebras Over the Injection Operad

In this appendix we identify the commutative monoids in the symmetric monoidal category of tame M-sets with the tame algebras over the injection operad, see Theorem A.13. We also show that for tame M-sets, the box product introduced in Definition 2.12 as an M-subset of the product is isomorphic to the operadic product, see Proposition A.17. The results in this appendix are combinatorial in nature and they are not needed for the homotopical analysis in the body of the paper; however, we feel that Theorem A.13 and Proposition A.17 are important to put the box product of M-spaces into context.

Construction A.1 (Injection operad). The injection monoid M is the monoid of 1-ary operations in the *injection operad* $\mathcal{M}$, an operad in the category of sets with respect to cartesian product. As before, we set $\mathbf{n} = \{1, \ldots, n\}$ and $\omega = \{1, 2, \ldots\}$. For $n \geq 0$ we let $M(n)$ denote the set of injective maps from the set $\mathbf{n} \times \omega$ to the set ω. The symmetric group Σ_n acts on $M(n)$ by permuting the 1st coordinate in $\mathbf{n} \times \omega$: for a permutation $\sigma \in \Sigma_n$ and an injection $\varphi \colon \mathbf{n} \times \omega \to \omega$ we set

$$(\varphi\sigma)(k, i) \;=\; \varphi(\sigma(k), i).$$

The collection of sets $\{M(n)\}_{n\geq 0}$ then becomes an operad $\mathcal{M}$ via 'disjoint union and composition'. More formally, the operad structure maps

$$\begin{aligned} M(k) \times M(n_1) \times \cdots \times M(n_k) &\to M(n_1 + \cdots + n_k) \\ (\varphi;\ \psi_1, \ldots, \psi_k) &\longmapsto \varphi \circ (\psi_1 + \cdots + \psi_k) \end{aligned} \tag{A.2}$$

are defined by setting

$$(\varphi \circ (\psi_1 + \cdots + \psi_k))(i,j) \ = \ \varphi(m,\ \psi_m(i - (n_1 + \cdots + n_{m-1}), j)),$$

where $m \in \{1, \ldots, k\}$ is the unique number such that $n_1 + \cdots + n_{m-1} < i \leq n_1 + \cdots + n_m$.

As for operads in any symmetric monoidal category, a categorical operad has a category of algebras over it.

Definition A.3. An $\mathcal{M}$-*set* is a set equipped with an algebra structure over the injection operad $\mathcal{M}$. A *morphism* of $\mathcal{M}$-sets is a morphism of algebras over $\mathcal{M}$.

Given an $\mathcal{M}$-set X, we write the action of the n-ary operations as

$$M(n) \times X^n \ \to \ X, \quad (\varphi, x_1, \ldots, x_n) \ \longmapsto \ \varphi_*(x_1, \ldots, x_n).$$

Because $M(1) = M$ is the injection monoid, every $\mathcal{M}$-set has an underlying M-set. We call an $\mathcal{M}$-set *tame* if the underlying M-set is tame in the sense of Definition 2.2. Because $\mathbf{0}$ is the empty set, the set $M(0)$ has a single element, the unique function $\emptyset = \mathbf{0} \times \omega \to \omega$. So every $\mathcal{M}$-set X has a distinguished element 0, the image of the action map $M(0) \to X$. The associativity of the operad action implies that the distinguished element 0 of an $\mathcal{M}$-set is supported on the empty set.

Example A.4. We let A be an abelian monoid. Then A becomes a "trivial" $\mathcal{M}$-set: for $n \geq 0$ and $\varphi \in M(n)$ we define

$$\varphi_* \, : \, A^n \ \to \ A, \quad \varphi_*(a_1, \ldots, a_n) \ = \ a_1 + \cdots + a_n$$

by summing in the monoid A; in particular, φ_* only depends on n, but not on φ. The $\mathcal{M}$-set associated to an abelian monoid has the special property that the monoid $M = M(1)$ acts trivially. One can show that $\mathcal{M}$-sets with trivial M-action "are" the abelian monoids.

The support of elements in sets with an action of $M = M(1)$ was introduced in Definition 2.4. Now we discuss the behavior of support in the more highly structured $\mathcal{M}$-sets, in particular its interaction with n-ary operations for $n \neq 1$. For example, we will show that given any pair of finitely supported elements x, y and an injection $\varphi \in M(2)$, then $\varphi_*(x, y)$ is finitely supported and

$$\mathrm{supp}(\varphi_*(x, y)) \ \subseteq \ \varphi(\{1\} \times \mathrm{supp}(x) \cup \{2\} \times \mathrm{supp}(y)).$$

For this we need the following lemma about the orbits of the right $M(1)^n$-action on the set $M(n)$. Since the monoid $M(1)^n$ is not a group, the relation resulting from this action is not symmetric and it is not a priori clear when two elements of $M(n)$ are equivalent in the equivalence relation that it generates. The following lemma is a discrete counterpart of the analogous property for the linear isometries operad, compare [6, Lemma I.8.1].

Lemma A.5. Let $n \geq 2$ and let $A_1, \dots, A_n$ be finite subsets ω. Consider the equivalence relation on the set $M(n)$ of injections from $\mathbf{n} \times \omega$ to ω generated by the relation

$$\varphi \ \sim \ \varphi(f_1 + \cdots + f_n)$$

for all $f_i \in M(1)$ such that f_i is the identity on A_i. Then two elements of $M(n)$ are equivalent if and only if they agree on the subsets $\{i\} \times A_i$ of $\mathbf{n} \times \omega$ for all $i = 1, \dots, n$.

Proof. The “only if'' is clear since the value of φ on $\{i\} \times A_i$ does not change when φ is modified by a generating relation. For the converse we consider $\varphi, \varphi' \in M(n)$, which agree on $\{i\} \times A_i$ for all $i = 1, \dots, n$. If we choose bijections between ω and the complements of the A_i in ω we can reduce (by conjugation with the bijections) to the special case where all A_i are empty.

We prove the special case by induction over n, starting with $n = 2$. We need to show that all injections in $M(2)$ are equivalent in the equivalence relation generated by the right action of $M(1)^2$. We show that an arbitrary injection φ is equivalent to the bijection $s \colon \mathbf{2} \times \omega \to \omega$ given by $s(1, i) = 2i - 1$ and $s(2, i) = 2i$.

Case 1: suppose that $\varphi(\{1\} \times \omega)$ consists entirely of odd numbers and $\varphi(\{2\} \times \omega)$ consists entirely of even numbers. We define $\alpha, \beta \in M$ by $\alpha(i) = (\varphi(1, i) + 1)/2$, respectively, $\beta(i) = \varphi(2, i)/2$. Then we have $\varphi = s(\alpha + \beta)$, so φ and s are equivalent.

Case 2: suppose that $\varphi(\{1\} \times \omega)$ consists entirely of even numbers and $\varphi(\{2\} \times \omega)$ consists entirely of odd numbers. We use the same kind of argument as in case 1.

Case 3: suppose that the image of φ consists entirely of odd numbers. We define $\psi \in M(2)$ by $\psi(1, i) = \varphi(1, i)$, $\psi(2, 2i) = \varphi(2, i)$ and $\psi(2, 2i - 1) = 2i$. Define $d_+, d_- \in M(1)$ by $d_+(i) = 2i$ and $d_-(i) = 2i - 1$. Then $\varphi = \psi(\mathrm{id} + d_+)$, so φ is equivalent to ψ, which in turn is equivalent to $\psi(\mathrm{id} + d_-)$. But $\psi(\mathrm{id} + d_-)$ satisfies the hypothesis of case 1, so altogether φ and s are equivalent.

Case 4: suppose that the image of φ consists entirely of even numbers. This can be reduced to case 2 by the analogous arguments as in case 3.

Case 5: In the general case we exploit that $\varphi(\{1\} \times \omega)$ contains infinitely many odd numbers or it contains infinitely many even numbers (or both). So we can choose an injection $\alpha \in M(1)$ such that $\varphi(\{1\} \times \alpha(\omega))$ consists of numbers of the same parity. Similarly we can choose $\beta \in M(1)$ such that $\varphi(\{2\} \times \beta(\omega))$ consists of numbers of the same parity. But then φ is equivalent to $\varphi(\alpha + \beta)$, which satisfies the hypothesis of one of the cases 1, 2, 3, or 4. So any $\varphi \in M(2)$ is equivalent to the elements s.

Now we perform the inductive step and suppose that $n \geq 2$. We let $\varphi, \psi \in M(n+1)$ be two injections. We let $s\colon \mathbf{2} \times \omega \to \omega$ be any bijection. By the inductive hypothesis, the injections

$$\varphi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \text{ and } \psi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \ \in \ M(n)$$

are equivalent under the action of $M(1)^n$ by precomposition. We may assume without loss of generality that

$$\varphi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \ = \ \psi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \circ (f_1 + \cdots + f_n)$$

for some $f_1, \ldots, f_n \in M(1)$. By the special case $n = 2$, the injections $f_n s$ and s in $M(2)$ are equivalent under the action of $M(1)^2$. We may assume without loss of generality that $f_n s = s(\alpha + \beta)$ for some $\alpha, \beta \in M(1)$. Then

$$\begin{aligned} \varphi \ &= \ \varphi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s) \\ &= \ \psi \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s^{-1}) \circ (f_1 + \cdots + f_n) \circ (\mathrm{id}_{\mathbf{n-1}\times\omega} + s) \\ &= \ \psi \circ (f_1 + \cdots + f_{n-1} + (s^{-1} f_n s)) \ = \ \psi \circ (f_1 + \cdots + f_{n-1} + \alpha + \beta). \end{aligned}$$

■

Proposition A.6. Let X be an $\mathcal{M}$-set, $x_i, \ldots, x_n$ finitely supported elements of X, and $\varphi, \psi \in M(n)$ for some $n \geq 1$.

(i) If φ and ψ agree on $\{i\} \times \mathrm{supp}(x_i)$ for all $i = 1, \ldots, n$, then $\varphi_*(x_1, \ldots, x_n) = \psi_*(x_1, \ldots, x_n)$.

(ii) The element $\varphi_*(x_1, \ldots, x_n)$ is finitely supported and

$$\mathrm{supp}(\varphi_*(x_1, \ldots, x_n)) \ \subseteq \bigcup_{i=1,\ldots,n} \varphi(\{i\} \times \mathrm{supp}(x_i)).$$

(iii) The subset X_τ of X consisting of finitely supported elements is closed under the action of the injection operad, and hence a tame $\mathcal{M}$-set.

Proof. (i) By Lemma A.5 we can assume without loss of generality that $\psi = \varphi(f_1 + \cdots + f_n)$ for suitable $f_1, \ldots, f_n \in M(1)$ such that f_i is the identity on $\mathrm{supp}(x_i)$. Then

$$\begin{aligned}\psi_*(x_1, \ldots, x_n) &= (\varphi(f_1 + \cdots + f_n))_*(x_1, \ldots, x_n) \\ &= \varphi_*((f_1)_*(x_1), \ldots, (f_n)_*(x_n)) = \varphi_*(x_1, \ldots, x_n).\end{aligned}$$

(ii) We let $u \in M(1)$ be an injection that fixes $\varphi(\{i\} \times \mathrm{supp}(x_i))$ elementwise for all $i = 1, \ldots, n$. Then $u\varphi$ agrees with φ on $\{i\} \times \mathrm{supp}(x_i)$ for all $i = 1, \ldots, n$. By Lemma A.5 we can assume without loss of generality that $u\varphi = \varphi(f_1 + \cdots + f_n)$ for suitable $f_1, \ldots, f_n \in M(1)$ such that f_i is the identity on $\mathrm{supp}(x_i)$. So

$$\begin{aligned}u_*(\varphi_*(x_1, \ldots, x_n)) &= (u\varphi)_*(x_1, \ldots, x_n) = (\varphi(f_1 + \cdots + f_n))_*(x_1, \ldots, x_n) \\ &= \varphi_*((f_1)_*(x_1), \ldots, (f_n)_*(x_n)) = \varphi_*(x_1, \ldots, x_n).\end{aligned}$$

This shows that $\varphi_*(x_1, \ldots, x_n)$ is supported on the union of the sets $\varphi(\{i\} \times \mathrm{supp}(x_i))$. Part (iii) is just another way to concisely summarize part (ii). ■

Construction A.7 (Box product of tame $\mathcal{M}$-sets). We let X and Y be tame $\mathcal{M}$-sets. The box product $X \boxtimes Y$ of the underlying M-sets was introduced in Definition 2.12. We claim that $X \boxtimes Y$ has a preferred $\mathcal{M}$-action, which makes it a coproduct in the category of tame $\mathcal{M}$-sets. Indeed, a general fact about algebras over an operad is that the product $X \times Y$ has a coordinatewise $\mathcal{M}$-action that makes it a product of X and Y in the category of $\mathcal{M}$-sets. Proposition A.6 (ii) shows that the subset $X \boxtimes Y$ of $X \times Y$ is invariant under the action of the full injection operad; hence, the product $\mathcal{M}$-action on $X \times Y$ restricts to an $\mathcal{M}$-action on $X \boxtimes Y$.

Given two morphisms between tame $\mathcal{M}$-sets $f: X \to X'$ and $g: Y \to Y'$, the map

$$f \boxtimes g: X \boxtimes Y \to X' \boxtimes Y'$$

is a morphism of $\mathcal{M}$-sets (and not just a morphism of M-sets). A formal consequence of the identification of tame $\mathcal{M}$-sets with commutative monoids under the box product in Theorem A.13 below is that the box product is actually a coproduct in the category of tame $\mathcal{M}$-sets.

Construction A.8 (Sum operation). We introduce a partially defined "sum" operation on a tame $\mathcal{M}$-set X; the operation is defined on pairs of elements with disjoint support, that is, it is a map

$$+: X \boxtimes X \to X.$$

Given two disjointly supported objects (x, y) of the tame $\mathcal{M}$-set X, we choose an injection $\varphi \colon \mathbf{2} \times \omega \to \omega$ such that $\varphi(1, j) = j$ for all $j \in \mathrm{supp}(x)$ and $\varphi(2, j) = j$ for all $j \in \mathrm{supp}(y)$. Then we define the sum of x and y as

$$x + y \;=\; \varphi_*(x, y). \tag{A.3}$$

Proposition A.6 implies that this is independent of the choice of φ, and that the support of $x + y$ satisfies

$$\mathrm{supp}(x + y) \;\subseteq\; \varphi(\{1\} \times \mathrm{supp}(x)) \;\cup\; \varphi(\{2\} \times \mathrm{supp}(y)) \;=\; \mathrm{supp}(x) \cup \mathrm{supp}(y). \tag{A.4}$$

Proposition A.11. Let X be a tame $\mathcal{M}$-set.

(i) For every pair (x, y) of disjointly supported elements of X, the relation

$$(x, y) \;=\; (x, 0) + (0, y)$$

holds in the $\mathcal{M}$-set $X \boxtimes X$.

(ii) The element 0 satisfies $x + 0 = x = 0 + x$ for all $x \in X$.

(iii) The sum map $+ \colon X \boxtimes X \to X$ is a morphism of $\mathcal{M}$-sets.

(iv) The relation $x + y = y + x$ holds for all disjointly supported elements x, y of X.

(v) The sum map is associative in the following sense: the relation

$$(x + y) + z \;=\; x + (y + z)$$

holds for every triple (x, y, z) of elements in X whose supports are pairwise disjoint.

Proof. (i) We choose $\varphi \in M(2)$ as in the definition of $x + y$, that is, $\varphi^1 = \varphi(1, -)$ is the identity on $\mathrm{supp}(x)$ and $\varphi^2 = \varphi(2, -)$ is the identity on $\mathrm{supp}(y)$. Since the distinguished object 0 has empty support, we have $\mathrm{supp}(x, 0) = \mathrm{supp}(x)$ and $\mathrm{supp}(0, y) = \mathrm{supp}(y)$. So φ can also be used to define $(x, 0) + (0, y)$. Hence,

$$(x, 0) + (0, y) = \varphi_*((x, 0), (0, y)) = (\varphi_*(x, 0), \varphi_*(0, y)) = (\varphi^1_*(x), \varphi^2_*(y)) = (x, y).$$

(ii) The distinguished 0 has empty support. We choose an injection $\varphi \in M(2)$ as in the definition of $x + 0$, that is, such that φ^1 is the identity on the support of x. Then

$$x + 0 \;=\; \varphi_*(x, 0) \;=\; \varphi^1_*(x) \;=\; x.$$

The relation $0 + x = x$ is proved in much the same way.

(iii) We must show that the sum map commutes with the action of n-ary operations in $M(n)$, for every $n \geq 0$. The case $n = 0$ is the relation $0 + 0 = 0$, which holds by (ii). For $n \geq 1$ we consider any $\lambda \in M(n)$, as well as elements (x_j, y_j) in $X \boxtimes X$ for $j = 1, \ldots, n$. We choose $\varphi_j \in M(2)$ as in the definition of $x_j + y_j$, that is, φ_j^1 is the identity on $\mathrm{supp}(x_j)$ and φ_j^2 is the identity on $\mathrm{supp}(y_j)$. We choose $\kappa \in M(2)$ as in the definition of $\lambda_*(x_1, \ldots, x_n) + \lambda_*(y_1, \ldots, y_n)$, that is, κ^1 is the identity on

$$\mathrm{supp}(\lambda_*(x_1, \ldots, x_n)) \subseteq \lambda\left(\bigcup_{j=1}^{n} \{j\} \times \mathrm{supp}(x_j)\right),$$

and κ^2 is the identity on $\mathrm{supp}(\lambda_*(y_1, \ldots, y_n))$. We let $\chi\colon \mathbf{2n} \to \mathbf{n2}$ be the shuffle permutation defined by $\chi(1, \ldots, 2n) = (1, n+1, 2, n+2, \ldots, 2n)$. Then the injections

$$\kappa(\lambda + \lambda) \quad \text{and} \quad \lambda(\varphi_1 + \cdots + \varphi_n)(\chi \times \mathrm{id}_\omega)$$

in $M(n+n)$ agree on the sets $\{j\} \times \mathrm{supp}(x_j)$ and $\{n+j\} \times \mathrm{supp}(y_j)$ for all $j = 1, \ldots, n$. So these two injections act in the same way on the $(n+n)$-tuple $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and we deduce the relations

$$\begin{aligned}
&\lambda_*(x_1, \ldots, x_n) + \lambda_*(y_1, \ldots, y_n) \\
&\quad = \kappa_*(\lambda_*(x_1, \ldots, x_n), \lambda_*(y_1, \ldots, y_n)) \\
&\quad = (\kappa(\lambda + \lambda))_*(x_1, \ldots, x_n, y_1, \ldots, y_n) \\
&\quad = (\lambda(\varphi_1 + \cdots + \varphi_n)(\chi \times \mathrm{id}_\omega))_*(x_1, \ldots, x_n, y_1, \ldots, y_n) \\
&\quad = (\lambda(\varphi_1 + \cdots + \varphi_n))_*(x_1, y_1, \ldots, x_n, y_n) \\
&\quad = \lambda_*((\varphi_1)_*(x_1, y_1), \ldots, (\varphi_n)_*(x_n, y_n)) \\
&\quad = \lambda_*(x_1 + y_1, \ldots, x_n + y_n).
\end{aligned}$$

(iv) We showed in part (iii) that the sum map $+\colon X \boxtimes X \to X$ is a morphism of $\mathcal{M}$-sets. Every morphism of $\mathcal{M}$-sets commutes with the sum operation, so the sum map $+\colon X \boxtimes X \to X$ takes sums in $X \boxtimes X$ to sums in X. In other words, the interchange relation

$$(x + y) + (y' + z) = (x + y') + (y + z) \tag{A.12}$$

holds for all $((x, y), (y', z))$ in $(X \boxtimes X) \boxtimes (X \boxtimes X)$. We specialize to the case where $x = z = 0$ are the distinguished elements. Since the distinguished element is a neutral element for the sum operation, we obtain the commutativity relation $y + y' = y' + y$.

(v) In the special case where $y' = 0$ is the neutral element, the interchange relation (A.12) becomes the associativity relation $(x + y) + z = x + (y + z)$. ■

Altogether this shows that for every tame $\mathcal{M}$-set X, the sum map $+\colon X \boxtimes X \to X$ is a unital, commutative and associative morphism of $\mathcal{M}$-sets. In particular, the sum map makes the underlying M-set of X into a commutative monoid with respect to the symmetric monoidal structure given by the box product. The next theorem shows that this data determines the $\mathcal{M}$-action completely, and tame $\mathcal{M}$-algebras are "the same as" commutative monoids in the symmetric monoidal category $(\mathrm{Set}^{M}_{\mathrm{tame}}, \boxtimes)$.

Theorem A.13. The functor

$$\mathrm{Set}^{\mathcal{M}}_{\mathrm{tame}} \to \mathit{Com}(\mathrm{Set}^{M}_{\mathrm{tame}}, \boxtimes), \quad X \longmapsto (X, +)$$

is an isomorphism of categories from the category of tame $\mathcal{M}$-sets to the category of commutative monoids in the symmetric monoidal category $(\mathrm{Set}^{M}_{\mathrm{tame}}, \boxtimes)$.

Proof. The functor is clearly faithful. Now we show that the functor is full. So we let $f\colon X \to Y$ be a morphism of commutative $\boxtimes$-monoids in tame M-sets; we must show that f is also a morphism of $\mathcal{M}$-sets. As a morphism of commutative $\boxtimes$-monoids, f in particular preserves the distinguished element 0 and is compatible with the action of $M = M(1)$.

To treat the case $n \geq 2$, we consider the map

$$\chi^n_X\colon M(n) \times X^n \to X^{\boxtimes n}, \quad (\lambda, x_1, \ldots, x_n) \mapsto (\lambda^1_*(x_1), \ldots, \lambda^n_*(x_n)),$$

where $\lambda^i = \lambda(i, -)$. Since χ^n_X is natural in X, the upper square in the following diagram commutes:

$$\begin{tikzcd}
M(n) \times X^n \arrow[r, "M(n)\times f^n"] \arrow[d, "\chi^n_X"'] & M(n) \times Y^n \arrow[d, "\chi^n_Y"] \\
X^{\boxtimes n} \arrow[r, "f^{\boxtimes n}"] \arrow[d, "\Sigma"'] & Y^{\boxtimes n} \arrow[d, "\Sigma"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd} \tag{A14}$$

Since f is a morphism of commutative $\boxtimes$-monoids, the lower square also commutes. We claim that the vertical composites are the operadic action maps. To show this we consider $(\lambda, x_1, \ldots, x_n) \in M(n) \times X^n$. The generalization of Proposition A.11 (i) to several components shows that

$$(x_1, \ldots, x_n) \;=\; \sum_{j=1}^{n} \iota_j(x_j),$$

where $\iota_j \colon X \to X^{\boxtimes n}$ puts x in the j-th slot and fills up the other coordinates with the distinguished element 0. The right-hand side of this equation is the sum for the $\mathcal{M}$-set $X^{\boxtimes n}$. We let $\varphi \in M(n)$ be an injection as in the definition of $\lambda^1_*(x_1) + \cdots + \lambda^n_*(x_n)$, that is, such that φ^j is the identity on $\lambda^j(\mathrm{supp}(x_j))$ for every $1 \le j \le n$. Then the injections $\varphi(\lambda^1 + \cdots + \lambda^n)$ and λ agree on $\{j\} \times \mathrm{supp}(x_j)$ for all $1 \le j \le n$ and we thus have $\lambda_*(\iota_j(x_j)) = (\varphi(\lambda^1 + \cdots + \lambda^n))_*(\iota_j(x_j)) = \lambda^j_*(x_j)$. As a consequence, we have

$$\lambda_*(x_1, \ldots, x_n) \;=\; \lambda_* \left(\sum_{j=1}^{n} \iota_j(x_j) \right) \;=\; \sum_{j=1}^{n} \lambda_* \left(\iota_j(x_j) \right) \;=\; \sum_{j=1}^{n} \lambda^j_*(x_j),$$

where the 2nd equation is the fact that the sum functor is a morphism of $\mathcal{M}$-sets, by Proposition A.11 (iii). Since the diagram (A.14) commutes, the map f is compatible with the operadic action of $M(n)$ for all $n \ge 0$, and so the functor is full.

The vertical factorization of the operadic action in (A.14) shows that it is determined by the M-action and the sum. Thus, the functor is injective on objects. It remains to show that it is surjective on objects, that is, that every commutative $\boxtimes$-monoid $(X, 0, +)$ arises from an $\mathcal{M}$-set through the sum construction. For $n \ge 0$, we define the operadic action map

$$M(n) \times X^n \;\to\; X \quad \text{by} \quad (\lambda, x_1, \ldots, x_n) \;\longmapsto\; \lambda_*(x_1, \ldots, x_n) \;=\; \sum_{j=1}^{n} \lambda^j_*(x_j).$$

This makes sense because the maps λ^j have disjoint images for $j = 1, \ldots, n$, so the elements $\lambda^1_*(x_1), \ldots, \lambda^n_*(x_1)$ can indeed be added. For $n = 0$, this definition returns the distinguished element 0, and for $n = 1$ it specializes to the given M-action. The operadic symmetry condition holds because the sum operation is commutative. The operadic associativity condition holds because the sum operation is associative and commutative. Finally, the sum map derived from this operadic action is the sum map we started out with, by definition. ∎

Now we recall the *operadic product*, another binary pairing for M-sets. We will show that the operadic product supports a natural map to the box product and that map is an isomorphism of M-sets whenever the factors are tame.

Construction A.15 (Operadic product). As before we denote by $M(2)$ the set of binary operations in the injection operad $\mathcal{M}$, that is, the set of injections from $\{1, 2\} \times \omega$ to ω.

As part of the operad structure of $\mathcal{M}$, the set $M(2)$ comes with a left M-action and a right M^2-action given by

$$M \times M(2) \times M^2 \ \to \ M(2), \qquad (f, \psi, (u, v)) \ \longmapsto \ f \circ \psi \circ (u + v).$$

Here $u + v\colon \{1, 2\} \times \omega \to \{1, 2\} \times \omega$ is defined by $(u + v)(1, i) = (1, u(i))$ and $(u + v)(2, i) = (2, v(i))$. Given two M-sets X and Y we can coequalize the right M^2-action on $M(2)$ with the left M^2-action on the product $X \times Y$ and form

$$M(2) \times_{M\times M} (X \times Y).$$

The left M-action on $M(2)$ by postcomposition descends to an M-action on this operadic product. Some care has to be taken when analyzing this construction: because the monoid M is not a group, it may be hard to figure out when two elements of $M(2) \times X \times Y$ become equal in the coequalizer. Viewed as a binary product on M-sets, $M(2) \times_{M\times M} (X \times Y)$ is coherently associative and commutative, but it does *not* have a unit object.

Now we let X and Y be tame M-sets. To state the next result, we write

$$p^1\colon M(2) \times_{M\times M} (X \times Y) \ \to \ X \quad \text{and} \quad p^2\colon M(2) \times_{M\times M} (X \times Y) \ \to \ Y$$

for the morphisms of M-sets defined by

$$p^1[\psi, x, y] \ = \ \psi^1 x \qquad \text{and} \qquad p^2[\psi, x, y] \ = \ \psi^2 y,$$

where $\psi^1 = \psi(1, -)$ and $\psi^2 = \psi(2, -)$. We observe that

$$\mathrm{supp}(p^1[\psi, x, y]) \ = \ \mathrm{supp}(\psi^1 x) \ \subseteq \ \psi^1(\mathrm{supp}(x)) \ = \ \psi(\{1\} \times \mathrm{supp}(x)),$$

and similarly $\mathrm{supp}(p^2[\psi, x, y]) \subseteq \psi(\{2\} \times \mathrm{supp}(y))$. Because ψ is injective, these two sets are disjoint. So all elements in the image of (p^1, p^2) lie in $X \boxtimes Y$. We write

$$\chi_{X,Y}\colon M(2) \times_{M\times M} (X \times Y) \ \to \ X \boxtimes Y \tag{A.15}$$

for the map (p^1, p^2) when we restrict the codomain to $X \boxtimes Y$.

Proposition A.17. For all tame M-sets X and Y, the map (A.15) is an isomorphism of M-sets.

Proof. Source and target of the morphism $\chi_{X,Y}$ commute with disjoint unions and orbits by group actions in each of the variables. So Theorem 2.11 reduces the claim to the special case $X = \mathcal{I}_m$ and $Y = \mathcal{I}_n$ for some $m, n \geq 0$. In this special case, the morphism $\chi_{\mathcal{I}_m,\mathcal{I}_n}$ factors as the following composite:

$$M(2) \times_{M\times M} (\mathcal{I}_m \times \mathcal{I}_n) \xrightarrow{q} M(2)/(M^{(m)} \times M^{(n)}) \xrightarrow{\epsilon} \mathcal{I}_{m+n} \underset{\cong}{\xrightarrow{\rho}} \mathcal{I}_m \boxtimes \mathcal{I}_m.$$

Here $M^{(m)}$ is the submonoid of M consisting of those injections that are the identity on $\{1,\ldots,m\}$. The 1st map q sends the class $[\psi, f, g]$ to the class of $\psi(f+g)$; it is an isomorphism because the map

$$M/M^{(m)} \to \mathcal{I}_m, \quad [f] \longmapsto f|_{\{1,\ldots,m\}}$$

is an isomorphism of M-sets. The 2nd map ϵ is defined by

$$\epsilon[\psi](i) = \begin{cases} \psi(1,i) & \text{for } 1 \leq i \leq m, \text{and} \\ \psi(2,i-m) & \text{for } m+1 \leq i \leq m+n. \end{cases} \tag{A.16}$$

The 2nd map ϵ is an isomorphism of M-sets by Lemma A.5, applied to $n = 2$, $A_1 = \{1,\ldots,m\}$ and $A_2 = \{1,\ldots,n\}$. The 3rd map ρ is the isomorphism discussed in Example 2.15. So the map $\chi_{\mathcal{I}_m,\mathcal{I}_n}$ is an isomorphism. ■

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