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The stable homotopy category is rigid

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The purpose of this paper is to prove that the stable homotopy category of algebraic topology is 'rigid' in the sense that it admits essentially only one model:

Rigidity Theorem. *Let $\mathcal{C}$ be a stable model category. If the homotopy category of $\mathcal{C}$ and the homotopy category of spectra are equivalent as triangulated categories, then there exists a Quillen equivalence between $\mathcal{C}$ and the model category of spectra.*

Our reference model is the category of spectra in the sense of Bousfield and Friedlander [BF, §2] with the stable model structure. The point of the rigidity theorem is that its hypotheses only refer to relatively little structure on the stable homotopy category, namely the suspension functor and the class of homotopy cofiber sequences. The conclusion is that all 'higher order structure' of stable homotopy theory is determined by these data. Examples of this higher order structure are the homotopy types of function spectra, which are not in general preserved by exact functors between triangulated categories, or the algebraic K-theory. However, the theorem does not claim that a model for the category of spectra can be constructed out of the triangulated homotopy category. Nor does it say that a given triangulated equivalence can be lifted to a Quillen equivalence of model categories.

The rigidity theorem completes a line of investigation begun by Brooke Shipley and the author in [SS] and improved 2-locally in [Sch]. We refer to those two papers for motivation, and for examples of triangulated categories that are not rigid, i.e., which admit exotic models. The new ingredients for the odd-primary case are roughly the following. The arguments of [Sch] reduce the problem at each prime p to a property of the first nonzero p-torsion class in the stable homotopy groups of spheres, which is the Hopf map η at the prime 2 and the class α_1 in the stable $(2p-3)$-stem for odd primes. At the prime 2 the Hopf map η is the reason that the mod-2 Moore spectrum fails to have a multiplication, even up to homotopy. At odd primes the mod-p Moore spectrum has a multiplication up to homotopy, but for $p = 3$ the class

α_1 is the obstruction to the multiplication being homotopy associative [To3, Lemma 6.2]. For primes $p \geq 5$ the multiplication of the mod-p Moore spectrum is homotopy associative and the relationship to the class α_1 is more subtle: α_1 shows up as the obstruction to an A_p-multiplication in the sense of Stasheff [St]. This fact is folklore, but I do not know a reference that uses the language of A_n-structures.

The rigidity theorem starts from a triangulated equivalence which is not assumed to be compatible with smash products in any sense. So the challenge is to bring the feature of α_1 as a coherence obstruction into a form that only refers to the triangulated structure. For this purpose we introduce the notion of a *k-coherent action* of a Moore space M on another object; see Definition 2.1. This concept is similar to Segal's approach to loop space structures via Δ-spaces (unpublished, but see [An, §5] and [Th, §1]). I expect that a k-coherent M-module is essentially the same as an A_k-action of the Moore space in the sense of Stasheff [St]. However, we do not use associahedra; the bookkeeping of higher coherence homotopies is done indirectly through extended powers of Moore spaces.

Organization of the paper. In Section 1 we recall the extended power construction and review some of its properties. Extended powers are used in Section 2 to define and study coherent actions of a mod-p Moore space on an object in a model category. Section 3 contains the main new result of this paper, Theorem 3.1; it says that in the situation of the rigidity theorem, the class α_1 acts nontrivially on the object in the homotopy category of $\mathcal{C}$ that corresponds to the sphere spectrum. Section 4 contains the proof of the rigidity theorem, which is a combination of Theorem 3.1 with the reduction arguments of [Sch].

While the rigidity theorem holds for general model categories, we restrict our attention to *simplicial* model categories in the body of the paper. We explain in Appendix A how the arguments have to be adapted in the general case. The author thinks that the necessary technicalities about framings can obstruct the flow of ideas, and that by deferring them to the appendix, the paper becomes easier to read.

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1. Extended powers

For our definition of a coherent action of the Moore space in Section 2 we need the extended power construction. In this section we recall this construction and review some of its properties. A key point is that 'small' extended powers of mod-p Moore spaces are again mod-p Moore spaces; see Lemma 1.4.

Definition 1.1. For a group G we denote by EG the nerve of the *transport category* with object set G and exactly one morphism between any ordered pair of objects. So EG is a *contractible simplicial set* with a free G-action. We are mainly interested in the case $G = \Sigma_n$, the symmetric group on n letters.

The n-th *extended power* of a pointed simplicial set X is defined as the homotopy orbit construction

$$D_n X \;=\; X^{\wedge n} \wedge_{\Sigma_n} E\Sigma_n^+ \;,$$

where the symmetric group Σ_n permutes the smash factors, and the '+' denotes a disjoint basepoint. We often identify the first extended power $D_1 X$ with X and use the convention $D_0 X = S^0$.

The injection $\Sigma_i \times \Sigma_j \longrightarrow \Sigma_{i+j}$ induces a $\Sigma_i \times \Sigma_j$-equivariant map of simplicial sets

$$E\Sigma_i \times E\Sigma_j \;\longrightarrow\; E\Sigma_{i+j}$$

and thus a map of extended powers

$$(1.2)\qquad \begin{aligned} \mu_{i,j} \;:\; D_i X \wedge D_j X \;\cong\; & X^{\wedge(i+j)} \wedge_{\Sigma_i \times \Sigma_j} (E\Sigma_i \times E\Sigma_j)^+ \\ & \longrightarrow X^{\wedge(i+j)} \wedge_{\Sigma_{i+j}} E\Sigma_{i+j}^+ = D_{i+j} X \;. \end{aligned}$$

We will refer to the maps $\mu_{i,j}$ as the *canonical maps* between extended powers. The canonical maps are associative in the sense that the following diagram commutes:

$$\begin{tikzcd} D_i X \wedge D_j X \wedge D_k X \arrow[r, "\mathrm{Id} \wedge \mu_{j,k}"] \arrow[d, "\mu_{i,j} \wedge \mathrm{Id}"'] & D_i X \wedge D_{j+k} X \arrow[d, "\mu_{i,j+k}"] \\ D_{i+j} X \wedge D_k X \arrow[r, "\mu_{i+j,k}"'] & D_{i+j+k} X \end{tikzcd}$$

for all $i, j, k \geq 0$. The canonical maps are also unital, so that after identifying $D_i X \wedge D_0 X$ and $D_0 X \wedge D_i X$ with $D_i X$ the maps $\mu_{i,0}$ and $\mu_{0,i}$ become the identity.

Throughout this paper, p denotes a prime number and M is a finite pointed simplicial set of the homotopy type of a mod-p Moore space with bottom cell in dimension 2. To be more specific, we define M as the pushout

$$\begin{tikzcd} \tilde{S}^2 \arrow[r] \arrow[d, "\times p"'] & C\tilde{S}^2 \arrow[d] \\ S^2 \arrow[r, "\iota"'] & M . \end{tikzcd} \tag{1.3}$$

Here $S^2 = \Delta[2]/\partial\Delta[2]$ is the 'small' simplicial model of the 2-sphere, $\tilde{S}^2$ is an appropriate subdivision of S^2 and $\times p : \tilde{S}^2 \longrightarrow S^2$ is a map of degree p. The simplicial set $C\tilde{S}^2 = \Delta[1] \wedge \tilde{S}^2$ is the cone on the subdivided sphere. The bottom cell inclusion $\iota : S^2 \longrightarrow M$ induces an epimorphism in integral homology in dimension 2.

With field coefficients, the reduced homology of the n-th extended power D_nX can be calculated as the homology of the symmetric group Σ_n with coefficients in the tensor power of the reduced homology of X. If X is a p-local space for a prime p strictly larger than n, then Σ_n has no higher group homology with these coefficients. This indicates a proof of the following lemma. The statement of Lemma 1.4 is not true for $n \geq p$, and D_pM is not a Moore space.

LEMMA 1.4. *Let p be an odd prime and let M denote the* mod-p *Moore space with bottom cell in dimension* 2 *defined above. Then for* $2 \leq n \leq p-1$ *the composite map*

$$S^2 \wedge D_{n-1}M \xrightarrow{\iota\wedge \mathrm{Id}} M \wedge D_{n-1}M \xrightarrow{\mu_{1,n-1}} D_nM$$

is a weak equivalence. Hence for such n the extended power D_nM is a mod-p *Moore space with bottom cell in dimension* $2n$.

In the rest of this section we discuss certain cubes of simplicial sets whose values are smash products of extended powers of M. These cubes and their colimits enter in Theorem 2.5 below, where we explain how the homotopy class α_1 is the obstruction to the existence of a p-coherent multiplication on the Moore space M.

For each $n \geq 1$ we define a certain $(n-1)$-dimensional cube $\mathcal{H}_n$ of pointed simplicial sets. In other words, $\mathcal{H}_n$ is a functor from the poset of subsets of the set $\{1, \ldots, n-1\}$, ordered under inclusion. For such a subset T we first define a subgroup $\Sigma(T)$ of the symmetric group Σ_n. The subgroup $\Sigma(T)$ consists of all those permutations that for all $i \notin T$ map the set $\{1, \ldots, i\}$ to itself. Thus if $1 \leq t_1 < t_2 < \cdots < t_j \leq n-1$ are the numbers that are *not* in T, then

$$\Sigma(T) \;\cong\; \Sigma_{t_1} \times \Sigma_{t_2-t_1} \times \cdots \times \Sigma_{n-t_j} \; . \tag{1.5}$$

For example, $\Sigma(\emptyset)$ is the trivial subgroup,

$$\Sigma(\{1, \ldots, i-1, i+1, \ldots, n-1\}) = \Sigma_i \times \Sigma_{n-i} \quad \text{and} \quad \Sigma(\{1, \ldots, n-1\}) = \Sigma_n \; .$$

The functor $\mathcal{H}_n$ sends a subset $T \subseteq \{1, \ldots, n-1\}$ to the homotopy orbit space

$$\mathcal{H}_n(T) = M^{\wedge n} \wedge_{\Sigma(T)} E\Sigma(T)^+ \ , \tag{1.6}$$

where $\Sigma(T)$ permutes the smash factors. For $S \subseteq T$ the group $\Sigma(S)$ is a subgroup of $\Sigma(T)$, so we get an induced map on homotopy orbits $\mathcal{H}_n(S) \longrightarrow \mathcal{H}_n(T)$ which makes $\mathcal{H}_n$ into a functor. Since the group $\Sigma(T)$ is a product of symmetric groups as in (1.5), the values $\mathcal{H}_n(T)$ are smash products of extended powers,

$$\mathcal{H}_n(T) \ \cong \ D_{t_1}M \wedge D_{t_2-t_1}M \wedge \cdots \wedge D_{n-t_j}M \ .$$

In this description, the map $\mathcal{H}_n(S) \longrightarrow \mathcal{H}_n(T)$ for $S \subseteq T$ is a smash product of canonical maps (1.2).

LEMMA 1.7. (a) *Each map $\mathcal{H}_n(S) \longrightarrow \mathcal{H}_n(T)$ in the cube $\mathcal{H}_n$ is injective, and for each pair of subsets $T, U \subseteq \{1, \ldots, n-1\}$ the simplicial set $\mathcal{H}_n(T \cap U)$ is the intersection of $\mathcal{H}_n(T)$ and $\mathcal{H}_n(U)$ in $\mathcal{H}_n(T \cup U)$. Thus the commutative square*

$$\begin{tikzcd}
\mathcal{H}_n(T \cap U) \arrow[r] \arrow[d] & \mathcal{H}_n(U) \arrow[d] \\
\mathcal{H}_n(T) \arrow[r] & \mathcal{H}_n(T \cup U)
\end{tikzcd}$$

is a pullback diagram.

(b) *For every subset T of $\{1, \ldots, n-1\}$, the natural map*

$$\mathrm{colim}_{S \subset T, S \neq T}\, \mathcal{H}_n(S) \ \longrightarrow \ \mathcal{H}_n(T)$$

from the colimit over the proper subsets of T to $\mathcal{H}_n(T)$ is injective.

Proof. (a) In order to show that $\mathcal{H}_n(S) \longrightarrow \mathcal{H}_n(T)$ is injective and that the square is a pullback we show these properties in each simplicial dimension k. The k-simplices of $\mathcal{H}_n(T)$ are given by

$$\begin{aligned}
\mathcal{H}_n(T)_k \ &= \ \left(M^{\wedge n} \wedge_{\Sigma(T)} E\Sigma(T)^+\right)_k \ = \ (M_k)^{\wedge n} \wedge_{\Sigma(T)} (\Sigma(T)^{k+1})^+ \\
&\cong \ (M_k)^{\wedge n} \wedge (\Sigma(T)^k)^+ \ \cong \ ((\bar{M}_k)^n \times \Sigma(T)^k)^+
\end{aligned}$$

where M_k denotes the pointed set of k-simplices of M, and $\bar{M}_k$ denotes the set of nonbasepoint k-simplices of M. The last two isomorphisms are not compatible with the simplicial structure, but that does not concern us. These isomorphisms are, however, natural for subgroups $\Sigma(T)$ of Σ_n. Since $\Sigma(S)$ is a subgroup of $\Sigma(T)$ for $S \subseteq T$, this description shows that the map $\mathcal{H}_n(S) \longrightarrow \mathcal{H}_n(T)$ is injective.

In order to show that the commutative square in question is a pullback, it suffices to show that the commutative square of groups

$$\begin{tikzcd}
\Sigma(T\cap U) \arrow[r] \arrow[d] & \Sigma(U) \arrow[d] \\
\Sigma(T) \arrow[r] & \Sigma(T\cup U)
\end{tikzcd}$$

is a pullback, because taking products and adding a disjoint basepoint preserve pullbacks. But this is a direct consequence of the definition of $\Sigma(T)$ as those permutations that stabilize the sets $\{1,\dots,i\}$ for all i not in T.

Property (b) is a consequence of (a), compare [Gw, Rm. 1.17]. □

We denote by H_n the colimit of the 'punctured cube', i.e., the restriction of $\mathcal{H}_n$ to the subposet consisting of all *proper* subsets of $\{1,\dots,n-1\}$ (i.e., strictly smaller than the whole set $\{1,\dots,n-1\}$, the empty subset is allowed here). For example, for $n=3$ the colimit H_3 is the pushout of the diagram

$$\begin{tikzcd}
\mathcal{H}_3(\{1\}) \arrow[d, equal] & \mathcal{H}_3(\emptyset) \arrow[l] \arrow[r] \arrow[d, equal] & \mathcal{H}_3(\{2\}) \arrow[d, equal] \\
D_2M\wedge M & M\wedge M\wedge M \arrow[l, "\mu_{1,1}\wedge \mathrm{Id}"'] \arrow[r, "\mathrm{Id}\wedge\mu_{1,1}"'] & M\wedge D_2M\,.
\end{tikzcd}$$

For $n=4$, the colimit H_4 is the colimit of the diagram

$$\begin{tikzcd}
M\wedge M\wedge M\wedge M \arrow[rr, "\mathrm{Id}\wedge\mathrm{Id}\wedge\mu_{1,1}"] \arrow[dr, "\mathrm{Id}\wedge\mu_{1,1}\wedge\mathrm{Id}"] \arrow[dd, "\mu_{1,1}\wedge\mathrm{Id}\wedge\mathrm{Id}"'] & & M\wedge M\wedge D_2M \arrow[dr, "\mathrm{Id}\wedge\mu_{1,2}"] \arrow[dd, "\mu_{1,1}\wedge\mathrm{Id}"] & \\
& M\wedge D_2M\wedge M \arrow[rr, "\mathrm{Id}\wedge\mu_{2,1}"'] \arrow[dd, "\mu_{1,2}\wedge\mathrm{Id}"] & & M\wedge D_3M \\
D_2M\wedge M\wedge M \arrow[rr, "\mathrm{Id}\wedge\mu_{1,1}"] \arrow[dr, "\mu_{2,1}\wedge\mathrm{Id}"'] & & D_2M\wedge D_2M & \\
& D_3M\wedge M & &
\end{tikzcd}$$

Part (ii) of the previous lemma says that $\mathcal{H}_n$ is a *cofibration cube* in the sense of [Gw, Def. 1.13]. As a consequence (see [Gw, Prop. 1.16]), the natural map from the *homotopy* colimit of the punctured cube to the (categorical) colimit H_n is a weak equivalence. Thus all colimits that occur in the following are actually homotopy colimits. The following lemma indicates that the colimit H_n of the punctured cube is weakly equivalent to the mapping cone of a map from a mod-p Moore space with bottom cell in dimension $4n-4$ to a mod-p Moore space with bottom cell in dimension $2n$.

LEMMA 1.8. *For $2 \leq n \leq p$, we consider the map $\gamma_n : S^2 \wedge D_{n-1}M \longrightarrow H_n$ defined as the composite*

$$(1.9) \qquad S^2 \wedge D_{n-1}M \xrightarrow{\iota \wedge \mathrm{Id}} M \wedge D_{n-1}M = \mathcal{H}_n\big(\{2, \ldots, n-1\}\big) \longrightarrow H_n \ .$$

Then γ_n is injective and its cofiber is weakly equivalent to a mod-p *Moore space with bottom cell in dimension $4n-3$.*

Proof. For $n = 2$ we have $\gamma_2 = \iota \wedge \mathrm{Id} : S^2 \wedge M \longrightarrow M \wedge M$ with cofiber $S^3 \wedge M$, which is indeed a Moore space with bottom cell in dimension 5. So we assume $n \geq 3$ and continue by induction.

We denote by $\kappa : H_{n-1} \longrightarrow D_{n-1}M = \mathcal{H}_{n-1}(\{1, \ldots, n-2\})$ the canonical map from the punctured colimit to the terminal vertex of the previous cube $\mathcal{H}_{n-1}$. The map κ is injective by Lemma 1.7 (b), and the map $\gamma_{n-1} : S^2 \wedge D_{n-2}M \longrightarrow H_{n-1}$ is a section up to homotopy to κ (the composite $\kappa \circ \gamma_{n-1}$ is a weak equivalence by Lemma 1.4). So the cofiber of κ is weakly equivalent to the suspension of the cofiber of γ_{n-1}, and thus, by induction, to a mod-p Moore space with bottom cell in dimension $4n-6$.

We let $\partial^0\mathcal{H}_n$ and $\partial^1\mathcal{H}_n$ denote the 'front' and 'back' face of the cube $\mathcal{H}_n$ with respect to the first coordinate. So $\partial^0\mathcal{H}_n$ and $\partial^1\mathcal{H}_n$ are the $(n-2)$-dimensional cubes indexed by subsets of $\{2, \ldots, n-1\}$ given by

$$(\partial^0\mathcal{H}_n)(T) = \mathcal{H}_n(T) \qquad \text{and} \qquad (\partial^1\mathcal{H}_n)(T) = \mathcal{H}_n(\{1\} \cup T)$$

for $T \subseteq \{2, \ldots, n-1\}$. We view the $(n-1)$-cube $\mathcal{H}_n$ as a morphism $\partial^0\mathcal{H}_n \longrightarrow \partial^1\mathcal{H}_n$ of $(n-2)$-cubes. We denote by $\mathcal{H}^+_{n-1}$ the previous cube $\mathcal{H}_{n-1}$, but indexed by subsets of $\{2, \ldots, n-1\}$ instead of subsets of $\{1, \ldots, n-2\}$, via the bijection $s : \{2, \ldots, n-1\} \longrightarrow \{1, \ldots, n-2\}$ that subtracts 1 from each element. In other words, we have

$$\mathcal{H}^+_{n-1}(T) \ = \ \mathcal{H}_{n-1}(s(T))$$

for $T \subseteq \{2, \ldots, n-1\}$. With this notation

$$(\partial^0\mathcal{H}_n)(T) = M \wedge \mathcal{H}^+_{n-1}(T)$$

as cubes indexed by subsets of $\{2, \ldots, n-1\}$.

By Lemma 1.4, the composite map of $(n-2)$-cubes

$$S^2 \wedge \mathcal{H}^+_{n-1} \xrightarrow{\iota\wedge\mathrm{Id}} M \wedge \mathcal{H}^+_{n-1} = \partial^0\mathcal{H}_n \xrightarrow{\mathcal{H}_n} \partial^1\mathcal{H}_n$$

is a weak equivalence at every $T \subseteq \{2, \ldots, n-1\}$. This implies that the induced map on *homotopy* colimits of punctured cubes is a weak equivalence. By Lemma 1.7 and the discussion thereafter, these homotopy colimits are weakly equivalent to the corresponding categorical colimits. Thus the composite map

$$S^2 \wedge H_{n-1} \xrightarrow{\iota\wedge\mathrm{Id}} M \wedge H_{n-1} \xrightarrow{\mathrm{colim}\,\mathcal{H}_n} |\partial^1\mathcal{H}_n|$$

is a weak equivalence, where $|\partial^1\mathcal{H}_n|$ is the colimit of the punctured $(n-2)$-cube $\partial^1\mathcal{H}_n$, i.e., the restriction of $\partial^1\mathcal{H}_n$ to the *proper* subsets of $\{2,\dots,n-1\}$. Note that the cubes $\mathcal{H}^+_{n-1}$ and $\mathcal{H}_{n-1}$ have the same punctured colimit, namely H_{n-1}.

We consider the following commutative diagram

$$\begin{tikzcd}
S^2 \wedge D_{n-1}M \arrow[d, "\iota\wedge\mathrm{Id}"'] & S^2 \wedge H_{n-1} \arrow[l, "\mathrm{Id}\wedge\kappa"'] \arrow[r, "\simeq"] \arrow[d, "\iota\wedge\mathrm{Id}"'] & |\partial^1\mathcal{H}_n| \arrow[d, equal] \\
M \wedge D_{n-1}M & M \wedge H_{n-1} \arrow[l, "\mathrm{Id}\wedge\kappa"] \arrow[r, "\mathrm{colim}\,\mathcal{H}_n"'] & |\partial^1\mathcal{H}_n|.
\end{tikzcd}$$

Here $\kappa : H_{n-1} \longrightarrow D_{n-1}M$ is the canonical map considered above, whose cofiber is a mod-p Moore space with bottom cell in dimension $4n-6$. The pushout of the lower row is the punctured colimit H_n and the map $\gamma_n : S^2 \wedge D_{n-1}M \longrightarrow H_n$ factors through the pushout of the upper row. Since the upper right horizontal map is a weak equivalence, the simplicial set $S^2 \wedge D_{n-1}M$ maps by a weak equivalence to the pushout of the upper row. So instead of γ_n we may study the map between the pushouts of the two rows.

Since colimits commute with each other, the vertical cofiber of the map between the horizontal pushouts is the pushout of the vertical cofibers, i.e., the pushout of the diagram

$$S^3 \wedge D_{n-1}M \xleftarrow{\mathrm{Id}\wedge\kappa} S^3 \wedge H_{n-1} \longrightarrow * \,.$$

As a threefold suspension of the cofiber of κ, this is indeed a mod-p Moore space with bottom cell in dimension $4n-3$. □

2. Coherent actions of Moore spaces

In this section we define coherent actions of a mod-p Moore space on an object in a model category, and we establish some elementary properties of this concept. We also show that the Moore space acts on itself in a tautological $(p-1)$-coherent fashion, and we prove that the homotopy class α_1 is the obstruction to extending this action to a p-coherent action.

We first restrict our attention to the class of *simplicial* model categories, where it makes sense to smash a pointed simplicial set, for example a Moore space, with an object of the category; this avoids a certain amount of technicalities. We indicate in Appendix A how the arguments have to be modified for general model categories without a simplicial structure.

As before, M denotes a certain finite pointed simplicial set of the homotopy type of a mod-p Moore space with bottom cell in dimension 2. A specific model was defined in (1.3). We denote by $\iota : S^2 = \Delta[2]/\partial\Delta[2] \longrightarrow M$ the 'bottom cell inclusion' from the small simplicial 2-sphere. The extended powers

$D_n M$ and the canonical maps $\mu_{i,j}$ were defined in 1.1 respectively (1.2). For $1 \leq n \leq p-1$ the extended power $D_n M$ is a mod-p Moore space with bottom cell in dimension $2n$, see Lemma 1.4.

Definition 2.1. Let $\mathcal{C}$ be a pointed simplicial model category, p a prime and $1 \leq k \leq p$. A *k-coherent M-module* X consists of a sequence

$$X_{(1)}, X_{(2)}, \ldots, X_{(k)}$$

of cofibrant objects of $\mathcal{C}$, together with morphisms in $\mathcal{C}$

$$\mu_{i,j} \ : \ D_i M \wedge X_{(j)} \ \longrightarrow \ X_{(i+j)}$$

for $1 \leq i, j$ and $i + j \leq k$, subject to the following two conditions.

- (Unitality) The composite

  $$S^2 \wedge X_{(j-1)} \xrightarrow{\iota \wedge \mathrm{Id}} M \wedge X_{(j-1)} \xrightarrow{\mu_{1,j-1}} X_{(j)}$$

  is a weak equivalence for each $2 \leq j \leq k$ (where we identify M with $D_1 M$).

- (Associativity) The square

  $$\begin{tikzcd} D_i M \wedge D_j M \wedge X_{(l)} \arrow[r, "\mathrm{Id} \wedge \mu_{j,l}"] \arrow[d, "\mu_{i,j} \wedge \mathrm{Id}"'] & D_i M \wedge X_{(j+l)} \arrow[d, "\mu_{i,j+l}"] \\ D_{i+j} M \wedge X_{(l)} \arrow[r, "\mu_{i+j,l}"'] & X_{(i+j+l)} \end{tikzcd}$$

  commutes for all $1 \leq i, j, l$ and $i + j + l \leq k$.

The *underlying object* of a k-coherent M-module X is the object $X_{(1)}$ of $\mathcal{C}$. We say that an object Y of $\mathcal{C}$ *admits a k-coherent M-action* if there exists a k-coherent M-module whose underlying $\mathcal{C}$-object is weakly equivalent to Y.

A *morphism* $f : X \longrightarrow Y$ of k-coherent M-modules consists of $\mathcal{C}$-morphisms $f_{(j)} : X_{(j)} \longrightarrow Y_{(j)}$ for $j = 1, \ldots, k$, such that the diagrams

$$\begin{tikzcd} D_i M \wedge X_{(j)} \arrow[r, "\mathrm{Id} \wedge f_{(j)}"] \arrow[d, "\mu_{i,j}"'] & D_i M \wedge Y_{(j)} \arrow[d, "\mu_{i,j}"] \\ X_{(i+j)} \arrow[r, "f_{(i+j)}"'] & Y_{(i+j)} \end{tikzcd}$$

commute for $1 \leq i, j$ and $i + j \leq k$.

Example 2.2. A 1-coherent M-module is just a cofibrant object with no further structure. If

$$X \ = \ \{X_{(1)}, \, X_{(2)}, \ \mu_{1,1} : M \wedge X_{(1)} \longrightarrow X_{(2)}\}$$

is a 2-coherent M-module, then in the homotopy category of $\mathcal{C}$ the composite map

$$\kappa \;:\; M \wedge X_{(1)} \xrightarrow{\mu_{1,1}} X_{(2)} \xleftarrow{\cong} S^2 \wedge X_{(1)}$$

is a retraction to $\iota \wedge \mathrm{Id} : S^2 \wedge X_{(1)} \longrightarrow M \wedge X_{(1)}$. So if the model category is stable, then the identity map of $X_{(1)}$ has order p in the group $[X_{(1)}, X_{(1)}]_{\mathrm{Ho}(\mathcal{C})}$. The converse is also true, but we do not need to know this. If the prime p is odd, then the 2-coherent M-action can be extended to a 3-coherent M-action if and only if the action map κ is homotopy associative.

Remark 2.3. The notion of a k-coherent M-module has a very similar flavor to Graeme Segal's concept of a *special* Δ-*space*, which encodes a loop space structure. A special Δ-space is a simplicial space X such that X_n is weakly equivalent to the n-fold cartesian product of X_1, in a specific way using the structure maps. The 'multiplication' in a Δ-space does not directly pair the underlying space X_1 with itself, but arises as one of the structure maps $X_2 \longrightarrow X_1$, when we keep in mind that X_2 is weakly equivalent to $X_1 \times X_1$. To my knowledge, Segal did not publish anything about special Δ-spaces, but they are discussed for example in [An, §5] and [Th, §1].

In a k-coherent M-module X, something similar is going on: an iterated application of the unitality condition shows that the i-th object $X_{(i)}$ of a k-coherent M-module X receives a weak equivalence from $S^{2i-2} \wedge X_{(1)}$. So up to these suspensions, all the objects $X_{(i)}$ that make up a coherent M-module are weakly equivalent. Thus the target of the action morphism $\mu_{1,1}$ is not the underlying object $X_{(1)}$, but something weakly equivalent to it (up to double suspension and in a specific way). So at the price of keeping different, but weakly equivalent (up to suspension), underlying objects around, we can get away with strictly associative actions.

We expect that a k-coherent M-module with underlying space Y corresponds to an A_k-action, á la Stasheff [St], of the Moore space on Y. In Stasheff's approach there is one fixed underlying object with a multiplication map which is only associative up to coherence homotopies parametrized by the associahedra. The standard way to endow the mod-p Moore spectrum with an A_{p-1}-multiplication is an (easy kind of) obstruction theory, using that the boundaries of the associahedra are spheres and that the p-local stable stems vanish in dimensions 1 through $2p-4$.

A convenient feature of our approach is that the Moore space is tautologically a $(p-1)$-coherent module over itself; the necessary data for the $(p-1)$-coherent action can simply and explicitly be written down using extended powers, as we spell out in the following example.

Example 2.4. The mod-p Moore space acts on itself in a $(p-1)$-coherent fashion, which we refer to as the *tautological* $(p-1)$-coherent M-module. We define a $(p-1)$-coherent M-module $\underline{M}$ by setting

$$\underline{M}_{(j)} \;=\; D_j M$$

for $1 \leq j \leq p-1$; in particular, the underlying object $\underline{M}_{(1)}$ is just the Moore space M. The action maps

$$\mu_{i,j} \;:\; D_i M \wedge \underline{M}_{(j)} \;=\; D_i M \wedge D_j M \;\longrightarrow\; D_{i+j} M \;=\; \underline{M}_{(i+j)}$$

are the canonical maps, in the sense of (1.2), between extended powers. The unit condition holds by Lemma 1.4. A futile attempt to extend $\underline{M}$ to a p-coherent M-module would be to define $\underline{M}_{(p)}$ as the p-th extended power $D_p M$. Then the associativity property still holds, but the map $S^2 \wedge D_{p-1} M \longrightarrow D_p M$ is not a weak equivalence.

Now suppose that Y is a cofibrant object of a pointed simplicial model category $\mathcal{C}$. Then we can define a tautological $(p-1)$-coherent M-action on $M \wedge Y$ by smashing the tautological module $\underline{M}$ with Y. More precisely, we define a $(p-1)$-coherent M-module $\underline{M \wedge Y}$ in $\mathcal{C}$ by

$$\underline{M \wedge Y}_{(j)} = \underline{M}_{(j)} \wedge Y = D_j M \wedge Y$$

for $j = 1, \ldots, p-1$, and similarly for the structure maps. The associativity and unitality conditions are inherited from $\underline{M}$.

For simplicity, we now restrict our attention to stable model categories, although the following theorem works in general. A pointed model category is *stable* if the suspension functor defined on its homotopy category is an equivalence. As usual we denote by $\alpha_1 : S^{2p} \longrightarrow S^3$ a generator of the p-primary part of the homotopy group $\pi_{2p} S^3$; see for example [Ra, Cor. 1.2.4]. Here $p = 2$ is allowed, and then α_1 is the suspension of the Hopf map $\eta : S^3 \longrightarrow S^2$. The next theorem says that the homotopy class α_1 'is' the obstruction to extending the tautological $(p-1)$-coherent M-module $\underline{M \wedge Y}$ to a p-coherent module.

THEOREM 2.5. *Let Y be a cofibrant object of a simplicial, stable model category $\mathcal{C}$ and let p be a prime. If the map*

$$\alpha_1 \wedge \mathrm{Id} \;:\; S^{2p} \wedge Y \longrightarrow S^3 \wedge Y$$

is trivial in the homotopy category of $\mathcal{C}$, then the tautological $(p-1)$-coherent M-action on $M \wedge Y$ defined in Example 2.4 can be extended to a p-coherent M-action.

Proof. We use the $(p-1)$-dimensional cube $\mathcal{H}_p$ of simplicial sets which was defined in (1.6), as well as H_p, the colimit of the 'punctured cube', i.e., the restriction of $\mathcal{H}_p$ to the subposet of all proper subsets of $\{1, \ldots, p-1\}$. The map $\gamma_p : S^2 \wedge D_{p-1} M \longrightarrow H_p$ is the composite

$$S^2 \wedge D_{p-1} M \;\xrightarrow{\iota \wedge \mathrm{Id}}\; M \wedge D_{p-1} M = \mathcal{H}_p\big(\{2, \ldots, p-1\}\big) \;\longrightarrow\; H_p \;.$$

Claim. An extension of $\underline{M \wedge Y}$ to a p-coherent M-action determines, and is determined by, a cofibrant object $\underline{M \wedge Y}_{(p)}$ of $\mathcal{C}$ and a morphism

$$\mu \ : \ H_p \wedge Y \longrightarrow \underline{M \wedge Y}_{(p)}$$

such that the composite $\mu \circ (\gamma_p \wedge \mathrm{Id}) : S^2 \wedge D_{p-1}M \wedge Y \longrightarrow \underline{M \wedge Y}_{(p)}$ is a weak equivalence.

We only need one direction of the claim, namely that from $\underline{M \wedge Y}_{(p)}$ and $\mu : H_p \wedge Y \longrightarrow \underline{M \wedge Y}_{(p)}$ as above we can build a p-coherent M-module extending $\underline{M \wedge Y}$: we simply define the additional action maps $\mu_{i,p-i}$ for $i = 1, \dots, p-1$ as the composites

$$D_i M \wedge \underline{M \wedge Y}_{(p-i)} = \mathcal{H}_p\big(\{1, \dots, \widehat{i}, \dots, p-1\}\big) \wedge Y \ \longrightarrow \ H_p \wedge Y \ \xrightarrow{\ \mu\ } \ \underline{M \wedge Y}_{(p)}$$

using the canonical map from the value of $\mathcal{H}_p$ at $\{1, \dots, \widehat{i}, \dots, p-1\}$ to the punctured colimit H_p. The associativity condition holds because these action maps factor through the punctured colimit H_p. The unitality condition holds since the composite

$$S^2 \wedge D_{p-1}M \wedge Y = S^2 \wedge \underline{M \wedge Y}_{(p-1)} \xrightarrow{\iota \wedge \mathrm{Id}} M \wedge \underline{M \wedge Y}_{(p-1)} \xrightarrow{\mu_{1,p-1}} \underline{M \wedge Y}_{(p)}$$

equals $\mu \circ (\gamma_p \wedge \mathrm{Id})$, which is a weak equivalence by assumption.

Since γ_p is an injective map of simplicial sets, the map

$$\gamma_p \wedge \mathrm{Id} \ : \ S^2 \wedge D_{p-1}M \wedge Y \longrightarrow H_p \wedge Y \tag{2.6}$$

is a cofibration between cofibrant objects in $\mathcal{C}$. By the claim it thus suffices to show that (2.6) admits a retraction in the homotopy category of $\mathcal{C}$. By Lemma 1.8, the cofiber of γ_p is a mod-p Moore space with bottom cell in dimension $4p-3$. So stably, γ_p sits in a cofiber sequence

$$M_{4p-4} \xrightarrow{\ f\ } M_{2p} \simeq S^2 \wedge D_{p-1}M \xrightarrow{\ \gamma_p\ } H_p \longrightarrow M_{4p-3} \tag{2.7}$$

for some stable map $f : M_{4p-4} \longrightarrow S^2 \wedge D_{p-1}M$; here the notation M_n refers to a mod-p Moore space with bottom cell in dimension n. The attaching map f is actually nonzero since the mod-p cohomology of H_p supports a nontrivial Steenrod operation P^1, but we do not need this.

The group $[M_{4p-4}, M_{2p}]^{\mathrm{stable}}$ of stable homotopy classes of maps which contain f is cyclic of order p generated by the composite

$$M_{4p-4} \xrightarrow{\ \text{pinch}\ } S^{4p-3} \xrightarrow{\ \Sigma^{2p-3}\alpha_1\ } S^{2p} \xrightarrow{\ \text{incl.}\ } M_{2p} \ .$$

If $\alpha_1 \wedge \mathrm{Id} : S^{2p} \wedge Y \longrightarrow S^3 \wedge Y$ is trivial in the homotopy category of $\mathcal{C}$, then so is the map $f \wedge \mathrm{Id} : M_{4p-4} \wedge Y \longrightarrow S^2 \wedge D_{p-1}M \wedge Y$. Thus the cofiber sequence (2.7) splits after smashing with Y, and so the map (2.6) admits a retraction in $\mathrm{Ho}(\mathcal{C})$. $\square$

The next lemma describes ways to make new coherent modules from old ones. By the *mapping cone* of a morphism $f : X \longrightarrow Y$ in a pointed simplicial model category we mean the object

$$C(f) \;=\; \Delta[1] \wedge X \cup_{1\times X} Y$$

where $\Delta[1]$ is pointed by the 0-vertex.

LEMMA 2.8. *Let $\mathcal{C}$ be a pointed simplicial model category.*

(a) *The mapping cone of any morphism of k-coherent M-modules has a natural structure of a k-coherent M-module.*

(b) *Let X be a k-coherent M-module for $2 \le k \le p$, and let $f : K \longrightarrow X_{(1)}$ be a morphism in $\mathcal{C}$ from a cofibrant object to the underlying object of X. Denote by $\widehat{f} : M \wedge K \longrightarrow X_{(2)}$ the* free extension *of f, i.e., the composite*

$$M \wedge K \xrightarrow{\mathrm{Id}\wedge f} M \wedge X_{(1)} \xrightarrow{\mu_{1,1}} X_{(2)} .$$

Then the mapping cone of the free extension $\widehat{f}$ admits a natural $(k-1)$-coherent M-action.

(c) *For any k-coherent M-module X there exists a morphism of k-coherent M-modules $\varphi : X \longrightarrow \bar{X}$ such that each component $\varphi_{(i)} : X_{(i)} \longrightarrow \bar{X}_{(i)}$ is an acyclic cofibration and such that the underlying object $\bar{X}_{(1)}$ is fibrant.*

Proof. (a) Let $f : X \longrightarrow Y$ be a morphism of k-coherent M-modules. For $1 \le j \le k$ we set $C(f)_{(j)} = C(f_{(j)})$, i.e., the j-th object $C(f)_{(j)}$ is the mapping cone of the map $f_{(j)} : X_{(j)} \longrightarrow Y_{(j)}$. The action map $\mu_{i,j}$ is obtained by passage to mapping cones in the horizontal direction in the commutative diagram

$$\begin{tikzcd} D_iM \wedge X_{(j)} \arrow[r, "\mathrm{Id}\wedge f_{(j)}"] \arrow[d, "\mu_{i,j}"'] & D_iM \wedge Y_{(j)} \arrow[d, "\mu_{i,j}"] \\ X_{(i+j)} \arrow[r, "f_{(i+j)}"'] & Y_{(i+j)} \end{tikzcd} .$$

Associativity is inherited from associativity of the actions on X and Y. Similarly, the unitality condition follows from the unitality of X and Y.

(b) Let us denote by $X_{\bullet+1}$ the $(k-1)$-coherent M-module obtained from X by forgetting the underlying space and reindexing the remaining parts, i.e., $(X_{\bullet+1})_{(i)} = X_{(i+1)}$ for $i = 1, \ldots, k-1$, and similarly for the action maps. As the bullet varies from 1 to $k-1$, the action maps of X constitute a morphism of $(k-1)$-coherent M-modules

$$\mu_{\bullet,1} \;:\; \underline{M} \wedge X_{(1)} \;\longrightarrow\; X_{\bullet+1} .$$

Here $\underline{M} \wedge X_{(1)}$ is the tautological coherent M-module of Example 2.4 (to be consistent with the notation of Example 2.4, we should underline all of $M \wedge X_{(1)}$, but that results in an awkward typesetting). The mapping cone of any morphism of $(k-1)$-coherent M-modules has a natural $(k-1)$-coherent M-action as in part (a). So we obtain the natural $(k-1)$-coherent M-module with underlying object $C(\widehat{f})$ as the mapping cone of the composite morphism

$$\underline{M} \wedge K \xrightarrow{\mathrm{Id} \wedge f} \underline{M} \wedge X_{(1)} \xrightarrow{\mu_{\bullet,1}} X_{(\bullet+1)} .$$

(c) We define the object $\bar{X}_{(1)}$ by choosing a fibrant replacement of $X_{(1)}$, i.e., an acyclic cofibration $\varphi_{(1)} : X_{(1)} \longrightarrow \bar{X}_{(1)}$ with fibrant target. For $2 \leq i \leq k$ we define the object $\bar{X}_{(i)}$ as the pushout

$$\begin{tikzcd} D_{i-1}M \wedge X_{(1)} \arrow[r, "\mathrm{Id} \wedge \varphi_{(1)}"] \arrow[d, "\mu_{i-1,1}"'] & D_{i-1}M \wedge \bar{X}_{(1)} \arrow[d, "\mu_{i-1,1}"] \\ X_{(i)} \arrow[r, "\varphi_{(i)}"'] & \bar{X}_{(i)} . \end{tikzcd}$$

As a cobase change of an acyclic cofibration, the morphism $\varphi_{(i)}$ is an acyclic cofibration. The structure maps $\mu_{i,j} : D_iM \wedge \bar{X}_{(j)} \longrightarrow \bar{X}_{(i+j)}$ are induced on pushouts by the commutative diagram

$$\begin{tikzcd} D_iM \wedge X_{(j)} \arrow[d, "\mu_{i,j}"'] & D_iM \wedge D_{j-1}M \wedge X_{(1)} \arrow[l, "\mathrm{Id} \wedge \mu_{j-1,1}"'] \arrow[r, "\mathrm{Id} \wedge \mathrm{Id} \wedge \varphi_{(1)}"] \arrow[d, "\mu_{i,j-1} \wedge \mathrm{Id}"] & D_iM \wedge D_{j-1}M \wedge \bar{X}_{(1)} \arrow[d, "\mu_{i,j-1} \wedge \mathrm{Id}"] \\ X_{(i+j)} & D_{i+j-1}M \wedge X_{(1)} \arrow[l, "\mu_{i-j-1,1}"] \arrow[r, "\mathrm{Id} \wedge \varphi_{(1)}"'] & D_{i+j-1}M \wedge \bar{X}_{(1)} \end{tikzcd}$$

The associativity and unitality conditions follow. □

3. Why α_1 acts nontrivially

The following Theorem 3.1 is the main technical result of this paper, and this entire section is devoted to its proof. In Section 4 we will prove the rigidity theorem by feeding Theorem 3.1 into the reduction arguments of [Sch]. Here we formulate and prove Theorem 3.1 for *simplicial* stable model categories only, but the result is true for all stable model categories. We explain in Appendix A how to modify the arguments for general model categories without a simplicial structure; the general form of Theorem 3.1 appears as Theorem A.1.

A pointed model category is *stable* if the suspension functor defined on its homotopy category is an equivalence. The homotopy category of a stable model category is naturally triangulated with suspension and cofibration sequences defining the shift operator and the distinguished triangles, compare [Ho, Prop. 7.1.6]. For any integer n, we denote the n-dimensional sphere spectrum by $\mathbf{S}^n$.

Theorem 3.1. *Let $\mathcal{C}$ be a simplicial stable model category and let*

$$\Phi \;:\; \mathrm{Ho}(\mathrm{Spectra}) \;\longrightarrow\; \mathrm{Ho}(\mathcal{C})$$

be an exact functor of triangulated categories that is fully faithful. Then for each odd prime p, the map

$$\alpha_1 \wedge \mathrm{Id} \;:\; S^{2p} \wedge \Phi(\mathbf{S}^0) \longrightarrow S^3 \wedge \Phi(\mathbf{S}^0)$$

is nontrivial in $\mathrm{Ho}(\mathcal{C})$, *where α_1 generates the p-primary part of the homotopy group $\pi_{2p}S^3$.*

A key point in Theorem 3.1 is that the functor Φ need not be compatible with smash products in any sense. Since Φ is faithful, the morphism $\Phi(\alpha_1 \wedge \mathrm{Id}) : \Phi(\mathbf{S}^{2p}) \longrightarrow \Phi(\mathbf{S}^3)$ is nontrivial, but that gives no *a priori* information about $\alpha_1 \wedge \mathrm{Id}_{\Phi(\mathbf{S}^0)}$.

The following proposition enters into the proof of Theorem 3.1. We continue to denote by M the mod-p Moore space with bottom cell in dimension 2 (or more precisely a specific finite pointed simplicial set defined in (1.3)). The inclusion $\iota : S^2 \longrightarrow M$ induces a morphism in the stable homotopy category $\iota \wedge \mathbf{S}^{n-2} : \mathbf{S}^n = S^2 \wedge \mathbf{S}^{n-2} \longrightarrow M \wedge \mathbf{S}^{n-2}$; by a slight abuse of notation, we denote this morphism by ι as well.

Proposition 3.2. *Let $\mathcal{C}$ be a simplicial stable model category and let*

$$\Phi \;:\; \mathrm{Ho}(\mathrm{Spectra}) \;\longrightarrow\; \mathrm{Ho}(\mathcal{C})$$

be an exact functor of triangulated categories that is fully faithful. Let

$$a \;:\; \mathbf{S}^n \;\longrightarrow\; E$$

be a morphism in the stable homotopy category and suppose that the j-fold suspension in $\mathrm{Ho}(\mathcal{C})$ *of the object $\Phi(E)$ admits a k-coherent M-action with $k \geq 2$.*

Then there exists an extension

$$\bar{a} \;:\; M \wedge \mathbf{S}^{n-2} \;\longrightarrow\; E$$

of a to the mod-p *Moore spectrum such that the $(2+j)$-fold suspension in* $\mathrm{Ho}(\mathcal{C})$ *of the object $\Phi(C(\bar{a}))$ admits a $(k-1)$-coherent M-action, where $C(\bar{a})$ is any mapping cone of $\bar{a}$.*

Proof. By assumption there exists a k-coherent M-module X in $\mathcal{C}$ and an isomorphism $X_{(1)} \cong S^j \wedge \Phi(E)$ in the homotopy category of $\mathcal{C}$, where $X_{(1)}$ is the underlying object of the coherent M-module X. By Lemma 2.8 (c) we can assume that $X_{(1)}$ is fibrant.

Let K be a cofibrant object of $\mathcal{C}$ which is isomorphic in the homotopy category to $S^j \wedge \Phi(\mathbf{S}^n)$. Since K is cofibrant and $X_{(1)}$ is fibrant, there is a morphism $f : K \longrightarrow X_{(1)}$ in the model category $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} S^j \wedge \Phi(\mathbf{S}^n) \arrow[r, "S^j\wedge\Phi(a)"] \arrow[d, "\cong"'] & S^j \wedge \Phi(E) \arrow[d, "\cong"] \\ K \arrow[r, "f"'] & X_{(1)} \end{tikzcd} \tag{3.3}$$

commutes in $\mathrm{Ho}(\mathcal{C})$. We consider the free extension, in the sense of Lemma 2.8 (b), of f. This free extension is a morphism $\widehat{f} : M \wedge K \longrightarrow X_{(2)}$ in $\mathcal{C}$, and part of a commutative square

$$\begin{tikzcd} S^2 \wedge K \arrow[r, "S^2\wedge f"] \arrow[d, "\iota\wedge K"'] & S^2 \wedge X_{(1)} \arrow[d, "\simeq"', "\mu_{1,1}\circ(\iota\wedge X_{(1)})"] \\ M \wedge K \arrow[r, "\widehat{f}"'] & X_{(2)}\,. \end{tikzcd} \tag{3.4}$$

We consider the following diagram in $\mathrm{Ho}(\mathcal{C})$,

(3.5)

$$\begin{tikzcd} S^{2+j} \wedge \Phi(\mathbf{S}^n) \arrow[r, "S^{2+j}\wedge\Phi(\iota)"'] \arrow[rr, bend left, "S^{2+j}\wedge\Phi(a)"] \arrow[dd, "\cong"'] & S^{2+j} \wedge \Phi(M \wedge \mathbf{S}^{n-2}) \arrow[r, dotted, "S^{2+j}\wedge\Phi(\bar{a})"'] \arrow[dd, dotted] & S^{2+j} \wedge \Phi(E) \arrow[d, "\cong"] \\ & & S^2 \wedge X_{(1)} \arrow[d, "\cong"] \\ S^2 \wedge K \arrow[r, "\iota\wedge K"'] & M \wedge K \arrow[r, "\widehat{f}"'] & X_{(2)}, \end{tikzcd}$$

in which the two dotted arrows have not yet been defined; the vertical maps decorated with isomorphism symbols are obtained from (3.3) by double suspension and from (3.4). The big outer diagram involving $S^{2+j} \wedge \Phi(a)$ commutes since the squares (3.3) and (3.4) do. There exists an isomorphism $S^{2+j} \wedge \Phi(M \wedge \mathbf{S}^{n-2}) \longrightarrow M \wedge K$ which makes the left square commute since both $S^{2+j} \wedge \Phi(M \wedge \mathbf{S}^{n-2})$ and $M \wedge K$ are cones of the degree p map of $S^{2+j} \wedge \Phi(\mathbf{S}^n)$. (More formally, we consider the diagram in $\mathrm{Ho}(\mathcal{C})$

$$\begin{tikzcd} S^{2+j} \wedge \Phi(\mathbf{S}^n) \arrow[r, "S^{2+j}\wedge\Phi(\cdot p)"] \arrow[d, "\cong"'] & S^{2+j} \wedge \Phi(\mathbf{S}^n) \arrow[r, "S^{2+j}\wedge\Phi(\iota)"] \arrow[d, "\cong"'] & S^{2+j} \wedge \Phi(M \wedge \mathbf{S}^{n-2}) \arrow[d, dotted] \\ S^2 \wedge K \arrow[r, "(\cdot p)\wedge K"'] & S^2 \wedge K \arrow[r, "\iota\wedge K"'] & M \wedge K \end{tikzcd}$$

both rows of which are part of distinguished triangles. Since the left hand square commutes, the axioms of a triangulated category allow us to choose a morphism in $\mathrm{Ho}(\mathcal{C})$ from $S^{2+j} \wedge \Phi(M \wedge \mathbf{S}^{n-2})$ to $M \wedge K$ making the right hand square commute, and this map is necessarily an isomorphism.)

Since Φ is full, there is a morphism $\bar{a} : M \wedge \mathbf{S}^{n-2} \longrightarrow E$ in the stable homotopy category such that $S^{2+j} \wedge \Phi(\bar{a})$ makes the right square in diagram (3.5) commute. Since vertical maps in (3.5) are isomorphisms, the two morphisms $\bar{a} \circ \iota, a : \mathbf{S}^n \longrightarrow E$ have the same image under Φ. Since Φ is faithful, the morphism $\bar{a}$ is thus an extension of $a : \mathbf{S}^n \longrightarrow E$.

Let $C(\bar{a})$ be any mapping cone of $\bar{a}$. Since Φ is an exact functor and the $(2+j)$-fold suspension of $\Phi(\bar{a})$ is isomorphic to the free extension $\widehat{f} : M \wedge K \longrightarrow X_{(2)}$, the object $S^{2+j} \wedge \Phi(C(\bar{a}))$ is isomorphic in $\mathrm{Ho}(\mathcal{C})$ to the mapping cone of $\widehat{f}$. Since X is a k-coherent M-module, that mapping cone admits a $(k-1)$-coherent M-action by Lemma 2.8 (b). □

Proof of Theorem 3.1. We argue by contradiction and suppose that there is an odd prime p for which $\alpha_1 \wedge \mathrm{Id}_{\Phi(\mathbf{S}^0)}$ is trivial. As usual we write $q = 2p-2$. We construct spectra E_i for $i = 0, \ldots, p-1$ with the following properties:

(a) the spectrum E_i has exactly one stable cell in dimensions jpq and $jpq+1$ for $j = 0, \ldots, i$, and no others;

(b) in the mod-p cohomology of E_i, the Steenrod operation P^{ip} is nontrivial from dimension 0 to dimension ipq;

(c) the $(2i+2)$-fold suspension in $\mathrm{Ho}(\mathcal{C})$ of the object $\Phi(E_i)$ admits a $(p-i)$-coherent M-action;

(d) there is a morphism in the stable homotopy category

$$a_i \ : \ \mathbf{S}^{(i+1)pq-1} \longrightarrow E_i$$

which is detected by the Steenrod operation P^p.

Property (d) means that in the mod-p cohomology of the mapping cone of a_i, the Steenrod operation P^p is nontrivial from dimension ipq to dimension $(i+1)pq$.

We start with $E_0 = M \wedge \mathbf{S}^{-2}$, a mod-$p$ Moore spectrum with bottom cell in dimension 0. Then properties (a) and (b) hold. Since α_1 acts trivially on $\Phi(\mathbf{S}^0)$ by assumption, Theorem 2.5 provides a p-coherent M-action on $M \wedge \Phi(\mathbf{S}^0)$. Since $M \wedge \Phi(\mathbf{S}^0)$ is isomorphic to $S^2 \wedge \Phi(M \wedge \mathbf{S}^{-2}) = S^2 \wedge \Phi(E_0)$ in $\mathrm{Ho}(\mathcal{C})$, this gives property (c).

We choose any morphism $a_0 : \mathbf{S}^{pq-1} \longrightarrow E_0$ which is detected by the operation P^p, for example the one constructed on page 60 of [To3, §5] (Toda denotes this morphism by $\tilde{\beta}_1$ since the composite with the pinch map $E_0 \longrightarrow \mathbf{S}^1$ is a unit multiple of the class β_1 which generates the p-component of the stable stem of dimension $pq-2$).

The construction of E_i for $1 \leq i \leq p-1$ is by induction on i. We extend the morphism $a_{i-1} : \mathbf{S}^{ipq-1} \longrightarrow E_{i-1}$ of the previous inductive step to a morphism $\bar{a}_{i-1} : M \wedge \mathbf{S}^{ipq-3} \longrightarrow E_{i-1}$ as in Proposition 3.2, and we let E_i be a mapping cone of the extension $\bar{a}_{i-1}$. Then property (a) for E_i follows from property (a) for E_{i-1}.

Since the attaching map $\bar{a}_{i-1} : M \wedge \mathbf{S}^{ipq-3} \longrightarrow E_{i-1}$ restricts to a_{i-1} on the bottom cell of the Moore spectrum, property (b) for E_{i-1} and property (d) for a_{i-1} show that the composite operation $P^p P^{(i-1)p}$ in the mod-p cohomology of E_i is nontrivial from dimension 0 to dimension ipq. The Adem relation

$$P^p P^{(i-1)p} \ = \ i \cdot P^{ip} \ + \ P^{ip-1} P^1 \tag{3.6}$$

holds for all positive i; since the operation P^1 acts trivially on the mod-p cohomology of E_i for dimensional reasons and since $i < p$, the operation P^{ip} is nontrivial on the bottom dimensional class of E_i, which means that property (b) holds. Property (c) is part of Proposition 3.2.

It remains to justify property (d) by constructing the map $a_i : \mathbf{S}^{(i+1)pq-1} \longrightarrow E_i$. The quotient of E_i by the subspectrum E_{i-1} is a Moore spectrum with bottom cell in dimension ipq. So we can start with the map

$$S^{ipq} \wedge a_0 \ : \ \mathbf{S}^{(i+1)pq-1} \longrightarrow S^{ipq} \wedge E_0 \simeq E_i / E_{i-1} \ ,$$

where a_0 was constructed in step (d) for $i = 0$. Since a_0 is detected by P^p, any lift in the stable homotopy category of $S^{ipq} \wedge a_0$ to E_i qualifies as the morphism a_i. The obstruction to lifting $S^{ipq} \wedge a_0$ to E_i is the composite in the stable homotopy category

$$\mathbf{S}^{(i+1)pq-1} \xrightarrow{S^{ipq} \wedge a_0} E_i / E_{i-1} \longrightarrow S^1 \wedge E_{i-1} \quad ,$$

where the second map is the boundary morphism. Since $S^1 \wedge E_{i-1}$ has stable cells in dimensions $jpq + 1$ and $jpq + 2$ for $j = 0, \ldots, i-1$, the obstructions lie in the p-local stable stems of dimension $jpq - 3$ and $jpq - 2$ for $j = 2, \ldots, i+1$; at this point we need some serious calculational input from stable homotopy theory.

Fact. For $j = 2, \ldots, p$, the p-components of the stable stems of dimension $jpq - 3$ and $jpq - 2$ are trivial.

To prove this fact we can appeal to the Adams-Novikov spectral sequence based on BP-homology. In [Ra, Thm. 4.4.20], Ravenel describes the E_2-term of this spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP\text{-comod}}(BP_*, BP_*) \quad \Longrightarrow \quad \mathbb{Z}_{(p)} \otimes \pi_{t-s}^{\text{stable}}$$

in the range of dimensions $t - s \leq (p^2 + p)q$, which is more than what we need. The only nontrivial classes with topological dimension of the form

$$t - s = jpq - 3 \qquad \text{or} \qquad t - s = jpq - 2 \quad \text{for } j = 2, \ldots, p$$

are the scalar multiples of

$$\alpha_1\beta_1^p \in E_2^{2p+1,(p^2+1)q} \qquad \text{and} \qquad \beta_{p/p} \in E_2^{2,p^2q} .$$

As a consequence of Toda's relation $\alpha_1\beta_1^p = 0$ for the corresponding homotopy classes (see [To1], [To2]), the class $\beta_{p/p}$ must annihilate $\alpha_1\beta_1^p$ by a d_{2p-1}-differential (compare [Ra, 4.4.22]). Thus there are no nontrivial infinite cycles in the dimensions we care about, and so there is no obstruction to lifting $S^{ipq} \wedge a_0$ to a morphism $a_i : \mathbf{S}^{(i+1)pq-1} \longrightarrow E_i$. Such a lift is not at all unique, but any lift is automatically detected by P^p. This finishes the inductive construction of the spectra E_i and the morphisms $a_i : \mathbf{S}^{(i+1)pq-1} \longrightarrow E_i$.

To finish the proof of Theorem 3.1 by contradiction, we consider the mapping cone $C(a_{p-1})$ of the last morphism $a_{p-1} : \mathbf{S}^{p^2q-1} \longrightarrow E_{p-1}$. By properties (b) and (d), the composite Steenrod operation

$$P^pP^{(p-1)p} \ : \ H^0(C(a_{p-1});\mathbb{F}_p) \ \longrightarrow \ H^{p^2q}(C(a_{p-1});\mathbb{F}_p)$$

is nontrivial. On the other hand, for $i = p$ the Adem relation (3.6) becomes $P^pP^{(p-1)p} = P^{p^2-1}P^1$. Since the operation P^1 is trivial in the cohomology of $C(a_{p-1})$ for dimensional reasons, we arrive at a contradiction, which means that $\alpha_1 \wedge \mathrm{Id}_{\Phi(\mathbf{S}^0)}$ is nontrivial. □

4. Proof of the rigidity theorem

In this final section we put all the pieces together to prove the rigidity theorem. We start with a criterion for when an endofunctor of the stable homotopy category is a self-equivalence. We continue to write $\mathbf{S}^n$ for the n-dimensional sphere spectrum and we let $\alpha_1 : \mathbf{S}^{2p-3} \longrightarrow \mathbf{S}^0$ generate the p-primary part of the stable stem of dimension $2p-3$, where p is an odd prime; this is the stable homotopy class represented by the unstable map $\alpha_1 : S^{2p} \longrightarrow S^3$ with the same name which shows up in Theorem 3.1.

Proposition 4.1. *Let F be an exact endofunctor of the stable homotopy category of spectra that preserves infinite sums and takes the sphere spectrum $\mathbf{S}^0$ to itself, up to isomorphism. If for every odd prime p the morphism $F(\alpha_1)$: $F(\mathbf{S}^{2p-3}) \longrightarrow F(\mathbf{S}^0)$ is nontrivial, then F is a self-equivalence.*

Proof. This proposition is essentially contained in [Sch], but there it is only stated locally at each prime separately. Since F is an exact functor that commutes with infinite coproducts, it also commutes with p-localization for every prime p; in other words, F takes p-local spectra to p-local spectra and for any spectrum Y, the map $F(Y) \longrightarrow F(Y_{(p)})$ extends uniquely to an isomorphism

$$F(Y)_{(p)} \longrightarrow F(Y_{(p)}) .$$

In order to show that F is an equivalence of categories, we may show separately for each prime p that the restriction of F to the category of p-local spectra

$$F \ : \ \mathrm{Ho}(\mathrm{Spectra}_{(p)}) \ \longrightarrow \ \mathrm{Ho}(\mathrm{Spectra}_{(p)})$$

is an equivalence. The functor F takes the p-local sphere to itself up to isomorphism, and thus it preserves the subcategory of *finite* p-local spectra (which coincide with the compact objects in $\mathrm{Ho}(\mathrm{Spectra}_{(p)})$). It suffices to show that F restricts to a self-equivalence of the category of finite p-local spectra (compare e.g. [Sch, Lemma 3.3]).

Proposition 3.1 of [Sch] reduces the problem further to showing that for $p = 2$, the Hopf maps η, ν and σ are in the image of F and for p odd, the class α_1 is in the image of F. Very roughly, this reduction makes precise the slogan that the homotopy groups of spheres are generated under higher order Toda brackets by the Hopf maps and the classes α_1 at odd primes. The precise argument, however, avoids explicit mentioning of Toda brackets, and is given in Section 4 of [Sch].

For an odd prime p, we assumed that $F(\alpha_1) : F(\mathbf{S}^{2p-3}) \longrightarrow F(\mathbf{S}^0)$ is nontrivial; since $F(\mathbf{S}^0)$ is isomorphic to $\mathbf{S}^0$, the map

$$F \ : \ [\mathbf{S}^{2p-3}, \mathbf{S}^0] \ \longrightarrow \ [F(\mathbf{S}^{2p-3}), F(\mathbf{S}^0)]$$

is an isomorphism on p-primary components, since these are cyclic of order p on both sides. Thus α_1 is in the image of F.

For $p = 2$, Proposition 3.2 of [Sch] says that the Hopf maps are automatically in the image of F. The key steps here are, again very roughly, as follows. Since F is exact and takes the sphere spectrum to itself, it also takes the mod-2 Moore spectrum to itself. Thus F cannot annihilate the degree 2 map on the mod-2 Moore spectrum. Since this degree 2 map factors over the Hopf map η, the functor F cannot annihilate η either, and must thus take η to itself. Using the relations $4\nu = \eta^3$ and $8\sigma \in \langle \nu, 8\iota, \nu \rangle$ one deduces from this that the Hopf maps ν and σ are also in the image of F. For the details of these arguments we refer to Section 5 of [Sch]. □

Proof of the rigidity theorem. We use the same kind of argument as in the 2-local situation considered in [Sch]; the key new ingredient is Theorem 3.1 (or rather Theorem A.1, the generalization to not necessarily simplicial model categories), which provides a handle on the element α_1 at odd primes.

The hypothesis of the rigidity theorem is that $\mathcal{C}$ is a stable model category and there exists an equivalence of triangulated categories

$$\Phi \ : \ \mathrm{Ho}(\mathrm{Spectra}) \ \longrightarrow \ \mathrm{Ho}(\mathcal{C})$$

from the stable homotopy category of spectra to the homotopy category of $\mathcal{C}$. We choose a cofibrant and fibrant object X of $\mathcal{C}$ that is isomorphic to $\Phi(\mathbf{S}^0)$ in the homotopy category of $\mathcal{C}$. By the universal property of the model category of spectra [SS, Thm 5.1], there is a left Quillen functor

$$- \wedge X \ : \ \text{Spectra} \longrightarrow \mathcal{C} \ ,$$

which takes the sphere spectrum $\mathbf{S}^0$ to X. We will show that the left derived functor

$$- \wedge^L X \ : \ \text{Ho(Spectra)} \longrightarrow \text{Ho}(\mathcal{C})$$

is an equivalence of categories. Then $- \wedge X$ and its right adjoint are in fact a Quillen equivalence. We warn the reader that our arguments do *not* show that the derived functor $- \wedge^L X$ is naturally isomorphic to Φ. So we do not answer the question whether the stable homotopy category admits exotic self-equivalences (other than shifts in either direction).

Since Φ is an equivalence, it suffices to show that the composite functor

$$F = \Phi^{-1} \circ (- \wedge^L X) \ : \ \text{Ho(Spectra)} \longrightarrow \text{Ho(Spectra)}$$

is an equivalence. Theorem 3.1 or Theorem A.1 shows that for each odd prime p, the map α_1 acts nontrivially on $\Phi(\mathbf{S}^0) \cong X$. Thus $F(\alpha_1) = \Phi^{-1}(\alpha_1 \wedge^L X)$ is also nontrivial and F is an equivalence of categories, by Proposition 4.1.

To be completely honest here we note that the expression '$\alpha_1 \wedge X$' has two different meanings. In Theorem 3.1 it refers to the action of the map of simplicial sets $\alpha_1 : S^{2p} \longrightarrow S^3$ on the object X. But we are interested in the value of the left Quillen functor $- \wedge X$: Spectra $\longrightarrow \mathcal{C}$ on the morphism of spectra $\alpha_1 : \mathbf{S}^{2p-3} \longrightarrow \mathbf{S}^0$. Since the 3-fold suspension of $\alpha_1 : \mathbf{S}^{2p-3} \longrightarrow \mathbf{S}^0$ is the effect of $\alpha_1 : S^{2p} \longrightarrow S^3$ on suspension spectra, these two uses of '$\alpha_1 \wedge X$' are consistent. This finishes the proof of the rigidity theorem. □

Essentially the same proof as above proves the following somewhat stronger form of the rigidity theorem. For the R-local model structure on spectra, see Section 4 of [SS].

THEOREM 4.2. *Let $\mathcal{C}$ be a stable model category whose homotopy category is compactly generated, and let R be a subring of the ring of rational numbers. Suppose that the full subcategory of compact objects in* $\text{Ho}(\mathcal{C})$ *is equivalent, as a triangulated category, to the homotopy category of finite R-local spectra. Then there exists a Quillen equivalence between $\mathcal{C}$ and the R-local model category of spectra whose left adjoint ends in $\mathcal{C}$.*

Besides the local form, the point of the stronger version is that already the subcategory of *finite* spectra determines the model category structure up to Quillen equivalence. In particular there is only one way to 'complete' the homotopy category of finite spectra to a triangulated category with infinite coproducts — as long as some underlying model structure exists. This gives a partial answer to Margolis' Uniqueness Conjecture [Ma, Ch. 2, §1] for the stable homotopy category.

Appendix A. Modifications in the absence of a simplicial structure

In the body of the paper, we defined coherent actions of a Moore space only for model categories that possess a compatible *simplicial* structure. In this appendix we develop coherent M-actions for general model categories, not necessarily simplicial, and we explain how the assumption of a simplicial structure can be dropped from the lemmas and propositions leading up to the rigidity theorem. The upshot is the following generalization of Theorem 3.1. The only difference between Theorems 3.1 and A.1 is that the hypothesis on $\mathcal{C}$ of being *simplicial* has been removed.

THEOREM A.1. *Let $\mathcal{C}$ be a stable model category and let*

$$\Phi \ : \ \mathrm{Ho}(\mathrm{Spectra}) \ \longrightarrow \ \mathrm{Ho}(\mathcal{C})$$

be an exact functor of triangulated categories that is fully faithful. Then for each odd prime p, the map

$$\alpha_1 \wedge \mathrm{Id} \ : \ S^{2p} \wedge \Phi(\mathbf{S}^0) \longrightarrow S^3 \wedge \Phi(\mathbf{S}^0)$$

is nontrivial in $\mathrm{Ho}(\mathcal{C})$, *where α_1 generates the p-primary part of the homotopy group $\pi_{2p}S^3$.*

To compensate for the lack of a simplicial structure, we work with *cosimplicial frames*. The theory of 'framings' of model categories goes back to Dwyer and Kan, who used the terminology *(co-)simplicial resolutions* [DK, 4.3]. We first try to motivate the definition of a cosimplicial frame: given a simplicial structure on a model category $\mathcal{C}$, any object X can be thickened up to a cosimplicial object by taking $\Delta[n] \times X$ as the object of n-cosimplices, where $\Delta[n]$ is the simplicial n-simplex and the product symbol refers to the given simplicial structure. If X is cofibrant, then this cosimplicial object, denoted by $\Delta[\bullet] \times X$, has two special properties:

(i) For every morphism $\alpha : [n] \longrightarrow [m]$ in the simplicial category Δ, the cosimplicial operator $\alpha_* : \Delta[n] \times X \longrightarrow \Delta[m] \times X$ is a weak equivalence in $\mathcal{C}$.

(ii) For all $m \geq 0$ the natural map

$$L_m(\Delta[\bullet] \times X) \ \longrightarrow \ \Delta[m] \times X$$

is a cofibration. Here L_m denotes the m-th *latching object* of a cosimplicial object, which we define below. The latching object can informally be described as the colimit of 'everything coming from below dimension m'.

In short, a cosimplicial frame is any cosimplicial object which behaves homotopically like $\Delta[\bullet] \times X$ in the sense that it has properties (i) and (ii) above.

After this motivation, we now discuss cosimplicial frames more formally. In what follows, $\mathcal{C}$ is a pointed model category and we denote by $c\mathcal{C}$ the category of cosimplicial objects in $\mathcal{C}$. In this pointed situation, products have to be replaced by smash products throughout. The *balanced smash product* (also known as the *coend*) $K \wedge_\Delta A$, of a pointed simplicial set K and a cosimplicial object A, is defined by

(A.2)

$$\begin{aligned} K \wedge_\Delta A &= \int^{n \in \Delta} K_n \wedge A^n \\ &= \text{coequalizer}\Big(\coprod_{\alpha:[n] \longrightarrow [m]} K_m \wedge A^n \underset{\mathrm{Id}\wedge\alpha_*}{\overset{\alpha^*\wedge \mathrm{Id}}{\rightrightarrows}} \coprod_{n \geq 0} K_n \wedge A^n \Big) . \end{aligned}$$

Here $K_n \wedge A^n$ denotes the coproduct of copies of A^n indexed by the set K_n, modulo the copy indexed by the basepoint of K_n. Then $\Delta[m]^+ \wedge_\Delta A$ is naturally isomorphic to the object of m-cosimplices of A. The object $L_m A = \partial\Delta[m]^+ \wedge_\Delta A$ is called the *m-th latching object* of A. The inclusion $\partial\Delta[m] \longrightarrow \Delta[m]$ induces a natural map $L_m A \longrightarrow A^m$.

Cosimplicial objects in any pointed model category admit the *Reedy model structure* in which the weak equivalences are the cosimplicial maps that are levelwise weak equivalences. A cosimplicial map $A \longrightarrow B$ is a *Reedy cofibration* if for all $m \geq 0$ the natural map

$$A^m \cup_{L_m A} L_m B \;\longrightarrow\; B^m$$

is a cofibration in $\mathcal{C}$. The Reedy fibrations are defined by the right lifting property for Reedy acyclic cofibrations or equivalently with the use of *matching objects*; see [Ho, 5.2.5] for more details. A *cosimplicial frame* (compare [Ho, 5.2.7]) is a cosimplicial object which is *homotopically constant* in the sense that each cosimplicial structure map is a weak equivalence in $\mathcal{C}$ (compare property (i) above) and Reedy cofibrant (compare property (ii)).

Note that the construction of $K \wedge_\Delta A$ does not need a simplicial structure on $\mathcal{C}$ and can be formed as soon as colimits exist in $\mathcal{C}$. If, however, the category $\mathcal{C}$ comes with a simplicial structure, then for every pointed simplicial set K, the object $K \wedge X$ given by the simplicial structure can be recovered from K and the special cosimplicial object $\Delta[\bullet]^+ \wedge X$ as

$$K \wedge X \;\cong\; K \wedge_\Delta (\Delta[\bullet]^+ \wedge X) .$$

Thus the m-th latching object of $\Delta[\bullet]^+ \wedge X$ is naturally isomorphic to $\partial\Delta[m]^+ \wedge X$. In the absence of a simplicial structure, a choice of cosimplicial frame $X^\bullet$ for an object X (i.e., with X^0 weakly equivalent to X) takes the role of $\Delta[\bullet]^+ \wedge X$. The smash product $K \wedge_\Delta X^\bullet$ then has all the relevant homotopical properties that the smash product $K \wedge X$ has in a pointed simplicial model category (but $K \wedge_\Delta X^\bullet$ is not functorial in X).

The construction $(K, A) \mapsto K \wedge_\Delta A$ which pairs a simplicial set and a cosimplicial object to an object in $\mathcal{C}$ can be extended to an action of pointed simplicial sets on the category $c\mathcal{C}$ of cosimplicial objects in $\mathcal{C}$. We define a new cosimplicial object $K \wedge A$ by

$$(K \wedge A)^n \;=\; (K \wedge \Delta[n]^+) \wedge_\Delta A\ .$$

Then we have a natural isomorphism $(K \wedge A)^0 \cong K \wedge_\Delta A$. If A is a cosimplicial frame and K a pointed simplicial set, then the cosimplicial object $K \wedge A$ is also a frame.

With the theory of cosimplicial frames at hand, we can now generalize the arguments in the body of this paper from simplicial to general model categories. The guiding principle is that cofibrant objects are replaced by cosimplicial frames, and fibrant objects are replaced by Reedy fibrant cosimplicial frames. For example, Definition 2.1 of a k-coherent M-module has to be modified as follows.

Definition A.3. Let $\mathcal{C}$ be a pointed model category, p a prime and $1 \leq k \leq p$. A *k-coherent M-module* consists of a sequence

$$X = X_{(1)}, X_{(2)}, \ldots, X_{(k)}$$

of cosimplicial frames in $\mathcal{C}$ together with morphisms in $c\mathcal{C}$

$$\mu_{i,j} \;:\; D_i M \wedge X_{(j)} \;\longrightarrow\; X_{(i+j)}$$

for $1 \leq i, j$ and $i + j \leq k$. The unitality condition now requires that the composite

$$S^2 \wedge X_{(j-1)} \xrightarrow{\iota \wedge \mathrm{Id}} M \wedge X_{(j-1)} \xrightarrow{\mu_{1,j-1}} X_{(j)}$$

is a level equivalence of cosimplicial objects for all $2 \leq j \leq k$. The associativity condition takes precisely the same form as in Definition 2.1, just that in the commuting square all the terms are now cosimplicial objects in $\mathcal{C}$. The *underlying object* of a k-coherent M-module X is the object $X^0_{(1)}$ of $\mathcal{C}$, i.e., the 0-cosimplices of $X_{(1)}$. We say that an object Y of $\mathcal{C}$ *admits a k-coherent M-action* if there exists a k-coherent M-module whose underlying $\mathcal{C}$-object is weakly equivalent to Y.

A *morphism* $f : X \longrightarrow Y$ of k-coherent M-modules consists of morphisms $f_{(j)} : X_{(j)} \longrightarrow Y_{(j)}$ of cosimplicial objects for $j = 1, \ldots, k$, such that the diagrams

$$\begin{tikzcd} D_iM \wedge X_{(j)} \arrow[r, "\mathrm{Id} \wedge f_{(j)}"] \arrow[d, "\mu_{i,j}"'] & D_iM \wedge Y_{(j)} \arrow[d, "\mu_{i,j}"] \\ X_{(i+j)} \arrow[r, "f_{(i+j)}"'] & Y_{(i+j)} \end{tikzcd}$$

commute for $1 \leq i, j$ and $i + j \leq k$.

If $\mathcal{C}$ is a pointed *simplicial* model category, then every k-coherent M-module in the sense of the earlier Definition 2.1 gives rise to a coherent M-module in the sense of Definition A.3, simply because the simplicial structure produces canonical frames: for a cofibrant object X, the cosimplicial object $\Delta[\bullet]^+ \wedge X$ is a cosimplicial frame.

For any cosimplicial frame Y in $\mathcal{C}$, the tautological $(p-1)$-coherent M-module $\underline{M}$ in the category of simplicial sets can be smashed termwise with Y to give a $(p-1)$-coherent M-module $\underline{M \wedge Y}$; compare Example 2.4. The underlying object $\underline{M \wedge Y}^0_{(1)} = M \wedge_\Delta Y$ is a model for the smash product in $\mathrm{Ho}(\mathcal{C})$ of the Moore space M with the object Y^0. The analog of Theorem 2.5 looks as follows.

Theorem A.4. *Let Y^0 be a cofibrant object of a stable model category $\mathcal{C}$ such that the map*

$$\alpha_1 \wedge \mathrm{Id} \ : \ S^{2p} \wedge Y^0 \longrightarrow S^3 \wedge Y^0$$

is trivial in the homotopy category of $\mathcal{C}$. Then for every cosimplicial frame Y which has Y^0 as its 0-cosimplices, the tautological $(p-1)$-coherent M-module $\underline{M \wedge Y}$ can be extended to a p-coherent M-module.

Morphisms between cosimplicial objects have mapping cones, so that the statement of part (a) of Lemma 2.8 does not change in the present more general context. The same arguments as in the proof of Lemma 2.8 also prove the following modified version.

Lemma A.5. *Let $\mathcal{C}$ be a pointed model category.*

(a) *The mapping cone of any morphism of k-coherent M-modules has a natural structure of a k-coherent M-module.*

(b) *Let X be a k-coherent M-module for $2 \leq k \leq p$, and let $f : K \longrightarrow X_{(1)}$ be a morphism of cosimplicial objects in $\mathcal{C}$ with source a cosimplicial frame. Denote by $\widehat{f} : M \wedge K \longrightarrow X_{(2)}$ the* free extension *of f, i.e., the composite*

$$M \wedge K \xrightarrow{\mathrm{Id} \wedge f} M \wedge X_{(1)} \xrightarrow{\mu_{1,1}} X_{(2)} \ .$$

Then the mapping cone of the free extension $\widehat{f}$ *admits a natural* $(k-1)$-*coherent* M-*action.*

(c) *For any* k-*coherent* M-*module* X *there exists a morphism of* k-*coherent* M-*modules* $\varphi : X \longrightarrow \bar{X}$ *such that each component* $\varphi_{(i)} : X_{(i)} \longrightarrow \bar{X}_{(i)}$ *is a Reedy acyclic cofibration and such that the cosimplicial object* $\bar{X}_{(1)}$ *is Reedy fibrant.*

With these modified definitions of coherent M-actions in place, the *proof of Theorem* A.1 is essentially the same as the proof of Theorem 3.1. First, one checks that Proposition 3.2 holds for general stable model categories, i.e., without the assumption that $\mathcal{C}$ is simplicial. In the proof, cofibrant objects are systematically replaced by cosimplicial frames, and fibrant objects by Reedy fibrant cosimplicial frames. The proof of Theorem A.1 is then literally the same as for Theorem 3.1. □

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