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Stable homotopy of algebraic theories

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Abstract

The simplicial objects in an algebraic category admit an abstract homotopy theory via a Quillen model category structure. We show that the associated stable homotopy theory is completely determined by a ring spectrum functorially associated with the algebraic theory. For several familiar algebraic theories we can identify the parameterizing ring spectrum; for other theories we obtain new examples of ring spectra. For the theory of commutative algebras we obtain a ring spectrum which is related to André–Quillen homology via certain spectral sequences. We show that the (co-)homology of an algebraic theory is isomorphic to the topological Hochschild (co-)homology of the parameterizing ring spectrum.

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The original motivation for this paper came from the attempt to generalize a rational result about the homotopy theory of commutative rings. For a map of commutative rings, Quillen [31] defined the cotangent complex as the left derived functor of abelianization; this construction is now referred to as André–Quillen homology. We wanted to obtain a topological variant of the cotangent complex by replacing "abelianization" by "stabilization", i.e., passage to spectra in the sense of stable homotopy theory. In [36] we made this precise by introducing a model category of spectra for simplicial commutative algebras. At the same time we showed that over the rational numbers, nothing really new is happening. More precisely, for a commutative $\mathbb{Q}$-algebra B, the stable homotopy theory of commutative simplicial B-algebras is equivalent to the homotopy

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theory of simplicial B-modules, see [36, Theorem 3.2.3]. Loosely speaking, stable homotopy and homology of commutative rings coincide rationally — just as they do for topological spaces.

One aim of this paper is to understand some of the torsion phenomena in the stable homotopy theory of commutative rings. The main structural result is again concerned with the stable homotopy category of commutative simplicial B-algebras, but where B is now an arbitrary commutative ring. We will see that this stable homotopy category is still a category of modules if one allows rings spectra rather than ordinary rings. For our purpose, the most convenient notion of ring spectrum is that of a *Gamma-ring* (see Definition 1.12). Gamma-rings are based on a symmetric monoidal smash product for Γ-spaces with good homotopical properties [9,25,38]. The homotopy theory of Gamma-rings and their modules is developed in [37]. The generalization of the rational result [36, Theorem 3.2.3] then reads:

Theorem. *Let B be a commutative ring. Then the stable homotopy theory of augmented commutative simplicial B-algebras is equivalent to the homotopy theory of modules over a certain Gamma-ring DB. The graded ring of homotopy groups of DB is isomorphic to the ring of stable homotopy operations of commutative augmented simplicial B-algebras. If B is a $\mathbb{Q}$-algebra, this DB is stably equivalent to the Eilenberg–MacLane Gamma-ring of B.*

This theorem is appropriately dealt with in a more general framework. We are led to consider pointed simplicial algebraic theories, or just simplicial theories for short. A simplicial theory T has a category of algebras; definitions will be given later, but thinking of a T-algebra as a simplicial set with certain algebraic operations and equational relations is a good guide line. Examples include simplicial sets, simplicial sets with an action of a group, (abelian) groups, modules over a simplicial ring, augmented (commutative) algebras over a commutative ring, Lie algebras and many more. The category of T-algebras is naturally a closed simplicial model category, thus allowing one to apply homotopy theoretic concepts. The category of *spectra* of T-algebras is also a closed simplicial model category, and its homotopy theory will be referred to as the stable homotopy theory of T. The above theorem then becomes a special case of Theorem 4.4, which in particular states.

Theorem. *To a simplicial theory T, there is functorially associated a Gamma-ring T^s. The graded ring of homotopy groups of T^s is isomorphic to the ring of stable homotopy operations of T-algebras. The stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over T^s.*

For several examples of algebraic theories the parameterizing Gamma-ring can be identified with something familiar: for the theory of sets we obtain the standard model of the sphere spectrum; the theories of monoids and groups give different, but stably equivalent models for the sphere spectrum; for sets with an action of a fixed group one gets the spherical group ring; the theory of modules over a fixed ring leads to the Eilenberg–MacLane Gamma-ring. More details on these examples can be found in Section 7; there we also list algebraic theories — such as the motivating example of commutative algebras — whose associated Gamma-rings give new homotopy types of ring spectra.

With the help of Theorem 4.4 we can deduce several structural properties that the homotopy theory of T-algebras shares with the ordinary homotopy theory of spaces. Among other things we provide Hurewicz and Whitehead theorems (Corollaries 5.3 and 5.4) as well as Atiyah–Hirzebruch and universal coefficient spectral sequences (see 5.5) which relate the Quillen homology of a T-algebra to its stable homotopy.

In [20, Section 4] Jibladze and Pirashvili defined the cohomology of a theory with coefficients in a functor that takes values in abelian group objects. We provide the link between the (co-)homology of an algebraic theory and its stable homotopy. Any coefficient functor G for the (co-)homology of the theory T gives rise to a bimodule $G^!$ over the Gamma-ring T^s. If T is the theory of modules over some ring, a theorem of Pirashvili and Waldhausen [30, Theorem 3.2] identifies theory homology with topological Hochschild homology (THH). In Theorem 6.7 we generalize this from rings to arbitrary algebraic theories and provide a cohomological analogue:

Theorem. *Let T be a pointed discrete algebraic theory and G a coefficient functor. Then there is a natural isomorphism*

$$\mathrm{H}_*(T;G) \cong \mathrm{THH}_*(T^s;G^!).$$

If G is additive, then there is a natural isomorphism

$$\mathrm{H}^*(T;G) \cong \mathrm{THH}^*(T^s;G^!).$$

The paper is organized as follows. In Section 1 we review Γ-spaces, Gamma-rings and their modules. Section 2 recalls algebraic theories. The unstable homotopy of algebraic theories is discussed in Section 3. Section 4 deals with the stable homotopy of an algebraic theory and proves the equivalence with modules over the associated Gamma-ring, Theorem 4.4. In Section 5 we describe the relationship between stable homotopy and Quillen homology for algebras over a theory. Section 6 establishes the equivalence of theory (co-)homology with topological Hochschild (co-)homology. Examples and applications are given in Section 7. The reader with little experience with the abstract notion of an algebraic theory might want to browse through the examples first, to get an idea of what the general theory is all about. We assume familiarity with the language of homotopical algebra [16,32].

1. Review of Γ-spaces and Gamma-rings

The category of Γ-spaces was introduced by Segal [38], who showed that it has a homotopy category equivalent to the stable homotopy category of connective spectra. Bousfield and Friedlander [9] considered a bigger category of Γ-spaces in which the ones introduced by Segal appeared as the *special* Γ-spaces (see Section 1.4). Their category admits a closed simplicial model category structure with a notion of stable weak equivalences giving rise again to the homotopy category of connective spectra. Then Lydakis [25] showed that Γ-spaces admit internal function objects and a symmetric monoidal smash product with good homotopical properties.

1.1. Γ-spaces. The category Γ^{op} is a skeletal category of the category of finite pointed sets. There is one object $n^+ = \{0, 1, \dots, n\}$ for every non-negative integer n, and morphisms are the maps of sets which send 0 to 0. Γ^{op} is equivalent to the opposite of Segal's category Γ [38]. A Γ-space is a covariant functor from Γ^{op} to the category of simplicial sets taking 0^+ to a one point simplicial set. A morphism of Γ-spaces is a natural transformation of functors. We denote the category of Γ-spaces by $\mathcal{GS}$. We sometimes need to talk about Γ-*sets*, by which we mean pointed functors from Γ^{op} to the category of pointed sets. Every Γ-space can be viewed as a simplicial object of Γ-sets. A symmetric monoidal smash product functor $\wedge : \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ is given by lexicographically ordering the elements of the set $n^+ \wedge m^+$. This smash product extends to a smash product for all pointed sets. We denote by $\mathbb{S}$ the Γ-space which takes n^+ to n^+, considered as a constant simplicial set. The spectrum associated to the Γ-space $\mathbb{S}$ (see 1.2) is the sphere spectrum. The representable Γ-spaces $\Gamma^n = \Gamma^{\mathrm{op}}(n^+, -)$ play a role analogous to that of the standard simplices in the category of simplicial sets. Γ^1 is isomorphic to $\mathbb{S}$. If X is a Γ-space and K a pointed simplicial set, a new Γ-space $X \wedge K$ is defined by setting $(X \wedge K)(n^+) = X(n^+) \wedge K$.

There are three kinds of hom objects for Γ-spaces X and Y. There is the set of morphisms (natural transformations) $\mathcal{GS}(X,Y)$. Then there is a simplicial hom set $\hom(X,Y)$, defined by

$$\hom(X,Y)_i = \mathcal{GS}(X \wedge (\Delta^i)^+, Y),$$

where the '$+$' denotes a disjoint basepoint. In this way $\mathcal{GS}$ becomes a simplicially enriched category. Finally there is an internal hom Γ-space $\mathrm{Hom}(X,Y)$ defined by

$$\mathrm{Hom}(X,Y)(n^+) = \hom(X, Y_{n^+\wedge}),$$

where $Y_{n^+\wedge}(m^+) = Y(n^+ \wedge m^+)$.

A Γ-space X can be prolonged, by direct limit, to a functor from the category of pointed sets to pointed simplicial sets. By degreewise evaluation and formation of the diagonal of the resulting bisimplicial sets, it can furthermore be promoted to a functor from the category of pointed simplicial sets to itself [9, Section 4]. This prolongation process has another description as the following coend [27, Section IX.6]. If X is a Γ-space and K a pointed simplicial set, the value of the extended functor on K is given by

$$\int^{+} \in \Gamma^{\mathrm{op}} K^n \wedge X(n^+).$$

The extended functor preserves weak equivalences of simplicial sets [BF, Proposition 4.9] and is automatically simplicial, i.e., it comes with coherent natural maps $K \wedge X(L) \to X(K \wedge L)$. We will not distinguish notationally between the prolonged functor and the original Γ-space.

1.2. Spectra. A spectrum X in the sense of [9, Definition 2.1] consists of a sequence of pointed simplicial sets X_n for $n \geqslant 0$, together with maps $S^1 \wedge X_n \to X_{n+1}$. A map of spectra $X \to Y$ consists of maps $X_n \to Y_n$ strictly commuting with the suspension maps. The homotopy groups of a spectrum X are defined as

$$\pi_n X = \mathrm{colim}_i\ \pi_{n+i}|X_i|.$$

A map of spectra is a *stable equivalence* if it induces isomorphisms on homotopy groups. A Γ-space X extends to a simplicial functor from all pointed simplicial sets, so it defines a spectrum $X(S)$ whose nth term is the value of the prolonged Γ-space at $S^n = S^1 \wedge \cdots \wedge S^1$ (n factors). For example, the Γ-space $\mathbb{S}$ becomes isomorphic to the identity functor of the category of pointed simplicial sets after prolongation. So the associated spectrum is given by $\mathbb{S}(S)_n = S^n$, i.e., $\mathbb{S}$ represents the sphere spectrum. The homotopy groups of a Γ-space are those of the associated spectrum, and they are always trivial in negative dimensions. A map of Γ-spaces is called a *stable equivalence* if it induces isomorphisms of homotopy groups.

1.3. Smash products. In [25, Theorem 2.2], Lydakis defines a smash product for Γ-spaces by the formula

$$(X \wedge Y)(n^+) = \operatorname{colim}_{k^+ \wedge l^+ \to n^+} \; X(k^+) \wedge Y(l^+).$$

The smash product is characterized by the universal property that Γ-space maps $X \wedge Y \to Z$ are in bijective correspondence with maps

$$X(k^+) \wedge Y(l^+) \to Z(k^+ \wedge l^+)$$

which are natural in both variables. By [25, Theorem 2.18], the smash product of Γ-spaces is associative and commutative with unit $\mathbb{S}$, up to coherent natural isomorphism. There is a natural isomorphism of Γ-spaces

$$\operatorname{Hom}(X \wedge Y, Z) \cong \operatorname{Hom}(X, \operatorname{Hom}(Y, Z)).$$

In other words, the category of Γ-spaces becomes a symmetric monoidal closed category.

1.4. Special Γ-spaces. A Γ-space X is called *special* if the map $X(k^+ \vee l^+) \to X(k^+) \times X(l^+)$ induced by the projections from $k^+ \vee l^+$ to k^+ and l^+ is a weak equivalence for all k and l. In this case, the weak map

$$X(1^+) \times X(1^+) \xleftarrow{\sim} X(2^+) \xrightarrow{X(\nabla)} X(1^+)$$

induces an abelian monoid structure on $\pi_0(X(1^+))$. Here $\nabla : 2^+ \to 1^+$ is the fold map defined by $\nabla(1) = 1 = \nabla(2)$. X is called *very special* if it is special and the monoid $\pi_0(X(1^+))$ is a group. By Segal's theorem ([38, Proposition 1.4], see also [9, Theorem 4.2]), the spectrum associated to a very special Γ-space X is an Ω-spectrum in the sense that the maps $|X(S^n)| \to \Omega|X(S^{n+1})|$ adjoint to the spectrum structure maps are homotopy equivalences. In particular, the homotopy groups of a very special Γ-space X are naturally isomorphic to the homotopy groups of the simplicial set $X(1^+)$.

1.5. Eilenberg–MacLane Γ-spaces. Simplicial abelian groups give rise to very special Γ-spaces via an Eilenberg–MacLane functor H. For a simplicial abelian group A, the Γ-space HA is defined by $(HA)(n^+) = A \otimes \tilde{\mathbb{Z}}[n^+]$ where $\tilde{\mathbb{Z}}[n^+]$ denotes the reduced free abelian group generated by the pointed set n^+. HA is very special and the associated spectrum is an Eilenberg–MacLane spectrum for A. The homotopy groups of HA are naturally isomorphic to the homotopy groups of A. The functor H embeds simplicial abelian groups as a full subcategory of $\mathcal{GS}$ and it has a left adjoint, left inverse functor L. For a Γ-space X, $L(X)$ is the cokernel of the map of simplicial abelian groups

$$(p_1)_* + (p_2)_* - \nabla_* : \tilde{\mathbb{Z}}[X(2^+)] \to \tilde{\mathbb{Z}}[X(1^+)].$$

Here p_1 and p_2 are the two projections from 2^+ to 1^+ in Γ^{op}. The functor L is compatible with the smash product of Γ-spaces (i.e., L is strong symmetric monoidal) and it preserves finite products, see [37, Lemma 1.2]. For Q-cofibrant Γ-spaces (see Section 1.6), L represents spectrum homology [37, Lemma 4.2].

1.6. Model category structures. Bousfield and Friedlander introduce two model category structures for Γ-spaces called the *strict* and the *stable* model categories [9, Sections 3.5 and 5.2]. It will be more convenient for our purposes to work with slightly different model category structures, which we call the Quillen- or Q-model category structures (see [37, Appendix A]). The strict and stable Q-structures have the same weak equivalences (hence the same homotopy categories) as the corresponding Bousfield–Friedlander model category structures, but different fibrations and cofibrations.

We call a map of Γ-spaces a strict Q-fibration (resp. a strict Q-equivalence) if it is a Kan fibration (resp. weak equivalence) of simplicial sets when evaluated at every $n^+ \in \Gamma^{op}$. Strict Q-cofibrations are defined as the maps having the left lifting property with respect to all strict acyclic Q-fibrations. The Q-cofibrations can be characterized in the spirit of [32, II.4, Remark 4] as the injective maps with projective cokernel, see [37, Lemma A.3(b)] for the precise statement. By [32, II.4, Theorem 4], the strict Q-notions of weak equivalences, fibrations and cofibrations make the category of Γ-spaces into a closed simplicial model category.

More important is the *stable* Q-model category structure. This one is obtained by localizing the strict Q-model category structure with respect to the stable equivalences. We call a map of Γ-spaces a stable Q-equivalence if it induces isomorphisms on homotopy groups. The stable Q-cofibrations are the strict Q-cofibrations and the stable Q-fibrations are defined by the right lifting property with respect to the stable acyclic Q-cofibrations. By [37, Theorem 1.5], these stable notions of Q-cofibrations, Q-fibrations and Q-equivalences make the category of Γ-spaces into a closed simplicial model category. A Γ-space X is stably Q-fibrant if and only if it is very special and $X(n^+)$ is fibrant as a simplicial set for all $n^+ \in \Gamma^{op}$. The Q-model category structure is compatible with Lydakis' smash product, see [37, Lemma 1.7]. For example, smashing with a Q-cofibrant Γ-space preserves stable equivalences.

Bousfield and Friedlander also introduce strict and stable model category structures for spectra. A map of spectra $X \to Y$ is a strict fibration (resp. strict weak equivalence) if all the maps $X_n \to Y_n$ are fibrations (resp. weak equivalences) of simplicial sets. It is a strict cofibration if $X_0 \to Y_0$ and

$$X_n \cup_{\Sigma X_{n-1}} \Sigma Y_{n-1} \to Y_n$$

(for $n \geqslant 1$) are cofibrations of simplicial sets. The stable weak equivalences are the maps which induce isomorphisms on homotopy groups. The stable cofibrations coincide with the strict cofibrations and the stable fibrations are the maps with the right lifting property with respect to stable acyclic cofibrations. There is a more explicit characterization of the stable fibrations in [9, Section 2.2]. In [9, Theorem 2.3] it is shown that the stable notions of fibrations, cofibrations and weak equivalences make the category of spectra into a closed simplicial model category. We show in Lemma A.3 that this model category structure is *cofibrantly generated* (see [15], [37, Definition A.2]).

1.7. Quillen equivalences. An adjoint functor pair between model categories is called a *Quillen pair* if the left adjoint L preserves cofibrations and acyclic cofibrations. An equivalent condition is to demand that the right adjoint R should preserve fibrations and acyclic fibrations. Under these conditions, the functors pass to an adjoint functor pair on homotopy categories (see [32, I.4, Theorem 3] or [16, Theorem 9.7(i)]). A Quillen functor pair is called a *Quillen equivalence* if the following condition holds: for every cofibrant object A of the source category of L and for every fibrant object X of the source category of R, a map $L(A) \to X$ is a weak equivalence if and only if its adjoint $A \to R(X)$ is a weak equivalence. Sometimes the right adjoint functor R preserves and detects all weak equivalences. Then the pair is a Quillen equivalence if for all cofibrant A the unit $A \to R(L(A))$ of the adjunction is a weak equivalence. A Quillen equivalence induces an equivalence of homotopy categories (see [32, I.4, Theorem 3] or [16, Theorem 9.7(ii)]), but it also preserves higher-order structure like (co-)fibration sequences, Toda brackets and the homotopy types of function complexes. If two model categories are related by a chain of Quillen equivalences, they can be viewed as having the same homotopy theory.

The functor that sends a Γ-space X to the spectrum $X(S)$ has a right adjoint [9, Lemma 4.6], and these two functors form a Quillen pair. One of the main theorems of [9] says that this Quillen pair induces an equivalence between the homotopy category of Γ-spaces, taken with respect to the stable structure of [9], and the stable homotopy category of connective spectra (see [9, Theorem 5.8]). Since every Q-cofibration is also a cofibration in the sense of Bousfield and Friedlander, and since the stable equivalences coincide in the two model category structures, the adjoint functor pair of [9, Lemma 4.6, Section 5] is also a Quillen pair with respect to the stable Q-model category structure for Γ-spaces.

1.8. The assembly map. Given two Γ-spaces X and Y, there is a natural map $X \wedge Y \to X \circ Y$ from the smash product to the composition product. The formal and homotopical properties of this *assembly map* are of fundamental importance to this paper. Since Γ-spaces prolong to functors defined on the category of pointed simplicial sets, they can be composed. Explicitly, for Γ-spaces X and Y, we set $(X \circ Y)(n^+) = X(Y(n^+))$. This composition $\circ$ is a monoidal (though not symmetric monoidal) product on the category of Γ-spaces. The unit is the same as for the smash product, it is the Γ-space $\mathbb{S}$ (alias Γ^1) which as a functor is the inclusion of Γ^{op} into all pointed simplicial sets.

The assembly map is obtained as follows. Prolonged Γ-spaces are naturally simplicial functors [9, Section 3], which means that there are natural coherent maps $X(K) \wedge L \to X(K \wedge L)$. This simplicial structure gives maps

$$X(n^+) \wedge Y(m^+) \to X(n^+ \wedge Y(m^+)) \to X(Y(n^+ \wedge m^+))$$

natural in both variables. From this the assembly map $X \wedge Y \to X \circ Y$ is obtained by the universal property of the smash product of Γ-spaces. The assembly map is associative and unital, $\mathbb{S}$ being the unit for both $\wedge$ and $\circ$. In technical terms: the identity functor on the category of Γ-spaces becomes a lax monoidal functor from $(\mathcal{GS}, \wedge)$ to $(\mathcal{GS}, \circ)$. The crucial homotopical property of the assembly map is that it is a stable equivalence whenever X or Y is cofibrant (see [25, Proposition 5.23]).

1.9. Stable excision. Every functor obtained from a Γ-space by prolongation preserves weak equivalences of simplicial sets and connectivity of maps [9, Sections 4.9 and 4.10]. But the homotopy functors arising as prolonged Γ-space have further connectivity and excision properties, such as for example the one we prove now.

1.10. Lemma. *Let F be a Γ-space and $X \to Y$ an injective map between simply connected pointed simplicial sets. Then the map*

$$F(Y)/F(X) \to F(Y/X)$$

is at least as connected as the suspension of $X \wedge (Y/X)$.

Proof. Every Γ-space admits a strict equivalence $F^c \to F$ from a Q-cofibrant Γ-space. Then the induced maps $F^c(X) \to F(X)$ are weak equivalences for all simplicial sets X, so we can assume that F is Q-cofibrant. If $F = \Gamma^n \wedge K$ for some simplicial set K, then $F(X) \cong X^n \wedge K$ and the lemma can be verified for F by inspection.

Now we consider an injection of Γ-spaces $F \to G$, we assume that the lemma holds for F and the quotient G/F, and we claim that the lemma follows for G. Since all spaces in sight are simply connected, it suffices to show that the homotopy cofiber (mapping cone) of the map $G(Y)/G(X) \to G(Y/X)$ is as connected as $\Sigma X \wedge (Y/X)$. We can calculate the total homotopy cofiber of the square

$$\begin{tikzcd}
F(Y)/F(X) \arrow[r] \arrow[d] & G(Y)/G(X) \arrow[d] \\
F(Y/X) \arrow[r] & G(Y/X)
\end{tikzcd}$$

in two ways. If we take horizontal homotopy cofibers first and use that the lemma holds for the Γ-space G/F, we conclude that the total homotopy cofiber is as connected as $\Sigma X \wedge (Y/X)$. By first taking vertical homotopy cofibers and using the lemma for F we see that the lemma holds for G.

We now know that if the lemma holds for a Γ-space F and if G is obtained from F by cobase change along one of the generating Q-cofibrations (see proof of [37, Theorem 1.5]) $\Gamma^n \wedge (\partial\Delta^i)^+ \to \Gamma^n \wedge (\Delta^i)^+$, then the lemma holds for G. Also if the lemma holds for all Γ-spaces in a (possibly transfinite) sequence of cofibrations, then it holds for the colimit of the sequence. Finally, the property we are interested in is preserved under retract. This finishes the proof since all Q-cofibrant Γ-spaces can be obtained from the trivial Γ-space by these operations (by the small object argument [37, Lemma A.1]). □

1.11. Gamma-rings and their modules. Our notion of ring spectrum is that of a *Gamma-ring*. Gamma-rings are the monoids in the symmetric monoidal category of Γ-spaces with respect to the smash product and they are special cases of "Functors with Smash Product" (FSPs, cf. [5, Section 1.1] or [30, Section 2.2]). One can describe Gamma-rings as "FSPs defined on finite sets". It was tempting to call these monoids "Γ-rings" but since that term should refer to a functor from Γ^{op} to the category of rings, the name "Gamma-ring" was chosen instead. A more detailed discussion of the homotopy theory of Gamma-rings can be found in [37].

1.12. Definition. A *Gamma-ring* is a monoid in the symmetric monoidal category of Γ-spaces with respect to the smash product. Explicitly, a Gamma-ring is a Γ-space R equipped with maps

$$\mathbb{S} \to R \quad \text{and} \quad R \wedge R \to R,$$

called the unit and multiplication map, which satisfy certain associativity and unit conditions (see [27, Section VII.3]). A Gamma-ring R is *commutative* if the multiplication map is unchanged when composed with the twist, or the symmetry isomorphism, of $R \wedge R$. A map of Gamma-rings is a map of Γ-spaces commuting with the multiplication and unit maps. If R is a Gamma-ring, a *left R-module* is a Γ-space N together with an action map $R \wedge N \to N$ satisfying associativity and unit conditions (see again [27, Section VII.4]). A map of left R-modules is a map of Γ-spaces commuting with the action of R. We denote the category of left R-modules by R-mod.

One similarly defines right modules. The unit $\mathbb{S}$ of the smash product is a Gamma-ring in a unique way. The category of $\mathbb{S}$-modules is isomorphic to the category of Γ-spaces. For a Gamma-ring R the opposite Gamma-ring R^{op} is defined by twisting the multiplication with the symmetry isomorphism of $R \wedge R$. Then the category of right R-modules is isomorphic to the category of left R^{op}-modules. The smash product of two Gamma-rings is naturally a Gamma-ring. n R-T-bimodule is defined to be a left $(R \wedge T^{\mathrm{op}})$-module. Because of the universal property of the smash product of Γ-spaces (see Section 1.3), Gamma-rings are in bijective correspondence with lax monoidal functors from the category Γ^{op} to the category of pointed simplicial sets (both under smash product). Similarly, commutative Gamma-rings correspond to lax symmetric monoidal functors.

Standard examples of Gamma-rings are monoid rings over the sphere Gamma-ring $\mathbb{S}$ and Eilenberg–MacLane models of classical rings. If M is a simplicial monoid, we define a Γ-space $\mathbb{S}[M]$ by

$$\mathbb{S}[M](n^+) = M^+ \wedge n^+$$

(so $\mathbb{S}[M]$ is isomorphic, as a Γ-space, to $\mathbb{S} \wedge M^+$). There is a unit map $\mathbb{S} \to \mathbb{S}[M]$ induced by the unit of M and a multiplication map $\mathbb{S}[M] \wedge \mathbb{S}[M] \to \mathbb{S}[M]$ induced by the multiplication of M which turn $\mathbb{S}[M]$ into a Gamma-ring. This construction of the monoid ring over $\mathbb{S}$ is left adjoint to the functor which takes a Gamma-ring R to the simplicial monoid $R(1^+)$.

If B is a simplicial ring, then the Eilenberg–MacLane Γ-space HB is naturally a Gamma-ring, simply because H is a lax monoidal functor [37, Lemma 1.2]. The functor H is full and faithful when considered as a functor from the category of simplicial rings to the category of Gamma-rings. The functor L is still left adjoint and left inverse to H. More examples of Gamma-rings arise from simplicial algebraic theories and as endomorphism Gamma-rings, see Sections 4.5 and 4.6 below.

Modules over a Gamma-ring form a model category. A map of R-modules is called a weak equivalence (resp. fibration) if it is a stable Q-equivalence (resp. stable Q-fibration) as a map of Γ-spaces. A map of R-modules is called a cofibration if it has the left lifting property with respect to all acyclic fibrations in R-mod. By [37, Theorem 2.2], the category of left R-modules becomes a closed simplicial model category this way. For a simplicial ring B, the functors H and L are a Quillen equivalence between the model category of HB-modules and the model category of simplicial B-modules [37, Theorem 4.4].

The category of R-modules inherits a smash product. More precisely, let M be a right R-module and N a left R-module. Then the smash product $M \wedge_R N$ is defined as the coequalizer, in the category of Γ-spaces, of the two maps $M \wedge R \wedge N \rightrightarrows M \wedge N$ induced by the action of R on M and

N respectively. If N is a cofibrant left R-module then the functor $- \wedge_R N$ takes stable equivalences of right R-modules to stable equivalences of Γ-spaces [37, Theorem 2.2]. We define the derived smash product $M \wedge^L_R N$ of M and N in the usual way: we choose a cofibrant left R-module N^c and a weak equivalence $N^c \xrightarrow{\sim} N$ and set $M \wedge^L_R N = M \wedge_R N^c$. The derived smash product is well defined up to stable equivalence. There are certain standard Tor spectral sequences converging to the homotopy groups of $M \wedge^L_R N$, see [37, Lemma 3.1].

2. Algebraic theories

Algebraic theories, introduced by Lawvere [23], formalize the concept of an algebraic object as a set together with n-ary operations for various $n \in \mathbb{N}$ and equational relations. A detailed exposition of algebraic theories can be found in [6, Section 3]. To do homotopy theory, we use algebraic theories which are enriched over the category of simplicial sets; these *simplicial theories* have been considered by Reedy [33]. The version of algebraic theories enriched over topological spaces can be found in [3, Eq. (6); 4, Chapter II] or [34,35]. For many purposes, topological and simplicial theories can be used interchangeably: the geometric realization and singular complex functors commute with finite products, so applying these to the simplicial hom set or the hom spaces respectively gets one from simplicial to topological theories and vice versa. We discuss examples of algebraic theories in Sections 2.6 and 7.

An algebraic theory is essentially a category with objects indexed by the natural numbers in such a way that the nth object is the n-fold product of the first object, in a specified way. The prototypical example (and in fact the initial algebraic theory) is the category Γ opposite to the category Γ^{op} of finite pointed sets.

2.1. Definition. A *simplicial theory* is a pointed simplicial category T together with a functor $\Gamma \to T$. It is required that T has the same discrete set of objects as Γ, and that $\Gamma \to T$ is the identity on objects and preserves products. A morphism of simplicial theories is a product preserving simplicial functor commuting with the functor from Γ.

One should think of the simplicial set $\hom_T(n^+, 1^+)$ as the space of n-ary operations in the theory T. If all the simplicial hom sets are discrete and if one omits the condition that T be pointed, one recovers the original definition of an algebraic theory. For emphasis we refer to such theories as *discrete* theories. Since morphisms of theories are always the identity on objects, a simplicial theory is the same as a simplicial object of pointed discrete theories. There is an initial simplicial theory, the theory of pointed sets. Formally it is the category Γ together with the identity functor.

2.2. Definition. If T is a simplicial theory, a T-*algebra* is a product preserving simplicial functor X from T to the category of pointed simplicial sets. A morphism of T-algebras is a natural transformation of functors. $X(1^+)$ is called the underlying simplicial set of the T-algebra X.

Note that even if T happens to be a discrete theory, T-algebras are still allowed to have a simplicial direction, unless otherwise stated. For a T-algebra X, each morphism $\varphi \in \hom_T(n^+, 1^+)_0$ gives rise to a map

$$X(1^+)^n \cong X(n^+) \xrightarrow{X(\varphi)} X(1^+).$$

This justifies thinking of a T-algebra as the underlying simplicial set together with n-ary operations parameterized by the simplicial set $\hom_T(n^+, 1^+)$. A morphism $\phi: R \to T$ of simplicial theories induces a functor $\phi^*: T\text{-alg} \to R\text{-alg}$ by precomposition with ϕ. The functor ϕ^* always has a left adjoint ϕ_* [6, Theorem 3.7.7].

2.3. Free *T*-algebras. The forgetful functor T-alg $\to$ (pt. simpl. sets), $X \mapsto X(1^+)$ has a left adjoint, the free T-algebra functor F^T. For a pointed simplicial set Y, the underlying simplicial set of $F^T(Y)$ is given by the coend

$$F^T(Y)(1^+) = \int^{n^+ \in \Gamma} Y^n \wedge \hom_T(n^+, 1^+).$$

(The forgetful functor is equal to the functor η^* for the unique theory morphism $\eta: \Gamma \to T$. Hence the free functor F^T is isomorphic to η_*.)

For any n, the representable functor $\hom_T(n^+, -)$ is a T-algebra isomorphic to the free T-algebra generated by n^+. In fact, this gives an equivalence of simplicial categories between T^{op} and the full subcategory of the finitely generated free T-algebras inside T-alg (cf. [6, Proposition 3.8.5]). The composite of the free T-algebra functor with the forgetful functor has the structure of a triple on the category of pointed simplicial sets. A triple also has a notion of algebras over it; however the category of algebras over the triple derived from T is equivalent to the category of T-algebras, so there is no ambiguity as to what a T-algebra is. Note that the triple, considered as a functor from the category of pointed simplicial sets to itself, is degreewise evaluable, i.e., it comes from a Γ-space.

This leads to an alternative characterization of algebraic theories via the free T-algebra functor: the category of simplicial theories is equivalent to the category of those triples on the category of pointed simplicial sets which are degreewise evaluable and commute with filtered colimits. Yet another way of putting this is as follows: we denote by T^s the restriction of the free T-algebra functor to the category Γ^{op}. So T^s is a Γ-space which comes with maps

$$T^s \circ T^s \to T^s \quad \text{and} \quad \mathbb{S} \to T^s$$

which are associative and unital. This just says that T^s becomes a monoid in the category of Γ-spaces with respect to the composition product of Section 1.8. The functor that sends a simplicial theory T to the Γ-space T^s with the composition product structure is an equivalence of categories

$$(\text{simplicial theories}) \cong (\text{monoids in } (\mathcal{GS}, \circ)).$$

This result can be found in [6, Proposition 4.6.2] for discrete theories, and in [4, Proposition 2.30] for topological theories.

2.4. Models. A T-algebra is also called a model of T in the category of pointed simplicial sets. We will also consider models of a theory in categories other than simplicial sets. Let $\mathscr{C}$ be a pointed category which has finite products and which is enriched over simplicial sets. A *model of T in $\mathscr{C}$* is a product preserving simplicial functor $X: T \to \mathscr{C}$. A morphism of models in $\mathscr{C}$ is a natural transformation of functors. The object $X(1^+) \in \mathscr{C}$ is called the underlying object of X. We will denote by $\mathscr{C}(T)$ the category of models of T in $\mathscr{C}$. For example, a group object of T-algebras is the same thing as a model of T in the category of simplicial groups. Group objects of T-algebras can also be viewed as models of the theory $\mathbb{G}$ of groups (see Section 2.6) in the category of T-algebras. In this paper we will consider models of T in the categories of simplicial abelian groups $\mathscr{Ab}(T)$, spectra $\mathscr{Sp}(T)$ and Γ-spaces $\mathscr{GS}(T)$.

We will later need the following lemma whose proof we omit.

2.5. Lemma. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories with finite products which are enriched over simplicial sets. Let $L: \mathscr{C} \to \mathscr{D}$ and $R: \mathscr{D} \to \mathscr{C}$ be a simplicial adjoint functor pair such that the left adjoint L preserves finite products. Then for any simplicial theory T, composition with L and R is an adjoint functor pair between the categories of models of $\mathscr{C}(T)$ and $\mathscr{D}(T)$.*

A first instance of this lemma is the geometric realization and singular complex functor pair between the categories of simplicial sets and compactly generated topological spaces. Another example to which we will apply the lemma is the adjoint functor pair H and L between Γ-spaces and simplicial abelian groups.

2.6. Example (The theory of groups). To illustrate the above definitions, we recall how the familiar example of the category of groups fits into the abstract framework. We denote the theory of groups by $\mathbb{G}$. This is a discrete theory with $\hom_{\mathbb{G}}(k^+, n^+)$ equal to the set of n-tuples of elements of the free group on k generators $\gamma_1, \ldots, \gamma_k$. The identity morphism of $\hom_{\mathbb{G}}(k^+, k^+)$ is the tuple $(\gamma_1, \ldots, \gamma_k)$ whose ith component is the word consisting only of the ith generator. Composition is given by substitution: if $(w_1, \ldots, w_n)$ and $(v_1, \ldots, v_k)$ are tuples of words in k resp. m generators, their composite is the tuple

$$(w_1(v_1, \ldots, v_k), \ldots, w_n(v_1, \ldots, v_k)).$$

Here $w_i(v_1, \ldots, v_k)$ means that in the word w_i each generator γ_j is substituted by the entire word v_j. The functor $\Gamma \to \mathbb{G}$ is given by

$$\hom_{\Gamma}(k^+, n^+) = \{0, 1, \ldots, k\}^n \to \hom_{\mathbb{G}}(k^+, n^+); \quad (i_1, \ldots, i_n) \mapsto (\gamma_{i_1}, \ldots, \gamma_{i_n})$$

with the convention $\gamma_0 = 1$. We claim that the category of $\mathbb{G}$-algebras is equivalent to the category of simplicial groups. So let $X: \mathbb{G} \to$ (pt. simpl. sets) be a $\mathbb{G}$-algebra, i.e., a product preserving functor. Then the underlying simplicial set $X(1^+)$ has a group structure as follows. The word $\gamma_1\gamma_2$ is an element of $\hom_{\mathbb{G}}(2^+, 1^+)$, so it gives rise to a multiplication map

$$\mu = X(\gamma_1\gamma_2): X(1^+)^2 \cong X(2^+) \to X(1^+).$$

The word γ_1^{-1} is an element of $\hom_{\mathbb{G}}(1^+, 1^+)$, so it gives rise to an inverse map

$$\iota = X(\gamma_1^{-1})\colon X(1^+) \to X(1^+).$$

The associativity and inverse conditions are codified in the category $\mathbb{G}$. We explain this for associativity. We consider the two elements $(\gamma_1\gamma_2, \gamma_3)$ and $(\gamma_1, \gamma_2\gamma_3)$ of $\mathrm{Hom}_{\mathbb{G}}(3^+, 2^+)$. Since multiplication in the free group on 3 generators is associative, the equality

$$(\gamma_1\gamma_2)\circ(\gamma_1\gamma_2, \gamma_3) = \gamma_1\gamma_2\gamma_3 = (\gamma_1\gamma_2)\circ(\gamma_1, \gamma_2\gamma_3)$$

holds in $\hom_{\mathbb{G}}(3^+, 1^+)$. The product preserving functor X takes this relation to the associativity condition $\mu\circ(\mu \times \mathrm{id}) = \mu\circ(\mathrm{id} \times \mu)$. Hence the underlying simplicial set of a $\mathbb{G}$-algebra is naturally a simplicial group.

For the converse let H be a simplicial group. Define a functor $\bar{H}$:$\mathbb{G} \to$ (pt. simpl. sets) on objects by $\bar{H}(n^+) = H^n$. The behavior on morphisms is again given by substitution: for $w = (w_1, \dots, w_n)$ from $\hom_{\mathbb{G}}(k^+, n^+)$, define

$$\bar{H}(w)(h_1, \dots, h_k) = (w_1(h_1, \dots, h_k), \dots, w_n(h_1, \dots, h_k)).$$

Here $w_i(h_1, \dots, h_k)$ means that the elements $h_j \in H$ are substituted for the generators γ_j into the word w_i, and then multiplication is carried out in the group H. We omit the verification that this $\bar{H}$ is a functor and that we in fact described an equivalence of categories

$$\mathbb{G}\text{-alg} \cong \text{(simplicial groups)}.$$

This example illustrates the general pattern: an arbitrary algebraic theory can be recovered from its category of algebras as the opposite of the full subcategory of finitely generated free objects. There is also a criterion for when a category with an adjoint functor pair to the category of sets is (equivalent to) the category of algebras over some algebraic theory, see [6, Theorem 3.9.1]. The example of the theory of groups is somewhat special because here the operations are generated by unary and binary operations, and all relations involved at most three generators. This need not be true for general theories. In fact, what makes Lawvere's notion of algebraic theories so elegant is that generating operations and relations are not mentioned at all. Instead, the category underlying the theory encodes *all* possible operations and their relations at the same time.

3. Unstable homotopy

The model category structure for algebras over a discrete theory is due to Quillen [32, II.4 Theorem 4]. For simplicial theories it was established by Reedy [33, Theorem I] and for topological theories by Schwänzl and Vogt [34, Theorem B]. A map of T-algebras is a weak equivalence or fibration if it is a weak equivalence or fibration on underlying simplicial sets respectively. A map of T-algebras is a cofibration if it has the left lifting property with respect to all acyclic fibrations. A map $A \to B$ of T-algebras is called a *free map* if there exists subsets $C_n \subset B_n$

which are stable under the simplicial degeneracy operators and such that in every simplicial dimension n, the induced map

$$A_n \amalg F^{T_n}(C_n) \to B_n$$

is an isomorphism of discrete T_n-algebras.

3.1 Theorem (Reedy [33, Theorem I]). *Let T be a simplicial theory. Then the category of T-algebras is a closed simplicial model category. The cofibrations are precisely the retracts of free maps.*

Proof. The model category structure follows from the lifting Lemma A.2, applied to the forgetful and free T-algebra functors between the categories of T-algebras and the category of pointed simplicial sets. The category of T-algebras is locally finitely presentable, see Lemma A.1. As the fibrant replacement functor Q we can take either Kan's functor Ex^∞ [21] or the composition of the singular complex and geometric realization functor (we then have to work in the category of compactly generated topological spaces). Each of these functors is simplicial and preserves finite products, so it passes to T-algebras. Quillen's argument [32, II.4, Remark 4] shows that the cofibrations are the retracts of the free maps. □

3.2. The bar resolution. It will be convenient to have at our disposal the standard cotriple resolution, also called bar resolution, for T-algebras. With the bar resolution, homotopical properties of free objects can be extended to all cofibrant objects. We denote by Φ^T the composite of the forgetful functor $T\text{-alg} \to (\text{pt. simpl. sets})$ with the free T-algebra functor $F^T : (\text{pt. simpl. sets}) \to T\text{-alg}$. This Φ^T is a cotriple on the category of T-algebras, so for any T-algebra X a simplicial object of T-algebras $\mathscr{B}(X)$ is defined by $\mathscr{B}(X)_n = (\Phi^T)^{n+1}(X)$ and with the usual simplicial face and degeneracy maps (see [28, Section 9.6]). By construction, $\mathscr{B}(X)_n$ is a free T-algebra and all structure maps except one of the face maps are free maps with respect to the defining generators. If Y_* is any simplicial object of T-algebras, then the diagonal of the underlying bisimplicial set is naturally another T-algebra. We denote this diagonal T-algebra by $|Y_*|$ and refer to it as the geometric realization. The bar resolution is augmented over the constant simplicial T-algebra X, so there is a map of T-algebras $|\mathscr{B}(X)| \to X$.

3.3. Lemma. *The augmentation map $|\mathscr{B}(X)| \to X$ is a weak equivalence of T-algebras. If $X \to Y$ is a map of T-algebras which is injective on underlying simplicial sets, then $|\mathscr{B}(X)| \to |\mathscr{B}(Y)|$ is a free map of T-algebras. In particular, $|\mathscr{B}(X)|$ is cofibrant as a T-algebra.*

Proof. The fact that the augmentation map $|\mathscr{B}(X)| \to X$ is a weak equivalence on underlying simplicial sets is a well known property of the bar construction, see e.g. [28, Proposition 9.8]. It remains to show that $|\mathscr{B}(-)|$ takes injective maps to free maps. In simplicial dimension n, the map $\mathscr{B}(X)_n \to \mathscr{B}(Y)_n$ is freely generated, in the category of discrete T_n-algebras, by the injective map underlying $(\Phi^T)^n(X)_n \to (\Phi^T)^n(Y)_n$. So if we let C_n denote the complement of the image of $(\Phi^T)^n(X)_n$ in $(\Phi^T)^n(Y)_n$, then the sets C_n satisfy the conditions in the definition of a free map. □

As an application of the bar resolution we obtain a homotopy invariance property, which can be found in [33, Corollary, p. 37]. In the context of topological theories, results of a similar kind can be found in [3, Theorem 8; 4, Theorem 4.58].

3.4. Lemma. *Let $\phi : T \to R$ be a morphism of simplicial theories which is a weak equivalence on all simplicial hom sets. Then the adjoint functors ϕ_* and ϕ^* are a Quillen equivalence between the model categories of T-algebras and R-algebras.*

Proof. ϕ^* preserves underlying simplicial sets, hence weak equivalences and fibrations, so ϕ^* and ϕ_* form a Quillen pair. Since ϕ^* detects and preserves all weak equivalences, it remains to check that for every cofibrant T-algebra X, the unit of the adjunction $X \to \phi^*\phi_*X$ is a weak equivalence. If X is freely generated by a finite set, this unit map is a weak equivalence by assumption. If X is freely generated by an arbitrary set, it is the filtered colimit, over cofibrations, of finitely generated free T-algebras. If X is freely generated by a simplicial set, the realization lemma (degreewise weak equivalences of bisimplicial sets induce weak equivalences on the diagonal simplicial sets, see [19, Proposition 2.4]) reduces to the discrete case. The bar resolution and the realization lemma reduce the general case to the free case. □

3.5. Homotopy fiber sequences. Since T-algebras have underlying simplicial sets, we can introduce the usual notions of connectivity of maps and objects. By the homotopy groups of a T-algebra we always mean the homotopy groups of the geometric realization of the underlying simplicial set. A T algebra will be called n-connected if all homotopy groups below and including dimension n are trivial. A map of T-algebras is called n-connected if it induces isomorphisms on homotopy groups below dimension n and an epimorphism in dimension n (for all choices of basepoint in the underlying simplicial set of the source). The *homotopy fiber* of a map of T-algebras $X \to Y$ is defined by choosing a factorization in the category of T-algebras

$$X \xrightarrow{\sim} W \twoheadrightarrow Y$$

of the map into a weak equivalence and a fibration and then taking the categorical fiber of the fibration $W \to Y$. The homotopy fiber is independent up to weak equivalence of the choice of factorization and it is also a homotopy fiber in the underlying category of simplicial sets. If $X \to Y \to Z$ are two maps of T-algebras whose composite is trivial, there is an induced map from X to the homotopy fiber of $Y \to Z$. We call the sequence $X \to Y \to Z$ an *n-homotopy fiber sequence* if the map $X \to$ hofiber$(Y \to Z)$ is n-connected.

3.6. Theorem. *Let $X \to Y$ be an $(n+k)$-connected cofibration between n-connected cofibrant T-algebras (with $n \geqslant 1$, $k \geqslant 0$). Then the cofibration sequence of T-algebras*

$$X \to Y \to Y//X$$

is a $(2n+k)$-homotopy fiber sequence. Here $Y//X$ denotes the quotient in the category of T-algebras which has to be distinguished from the quotient of the underlying simplicial sets.

Proof. We first prove the theorem in the special case where the map $X \to Y$ is obtained from an $(n+k)$-connected cofibration $A \to B$ between n-connected simplicial sets by application of the free T-algebra functor. In this case the quotient of T-algebras $Y//X$ is isomorphic to the free T-algebra generated by the quotient of simplicial sets B/A. The free T-algebras functor is a prolonged Γ-space, so by Lemma 1.10 the map

$$Y/X = T(B)/T(A) \to T(B/A) = Y//X$$

is $(2n+k+2)$-connected. By the Blakers–Massey homotopy excision theorem, the sequence $X \to Y \to Y/X$ is a $(2n+k)$-homotopy fiber sequence of simplicial sets. The theorem follows in the special case.

In the general case we use the bar resolution. The cofibration sequence of T-algebras $X \to Y \to Y//X$ admits a map from the cofibration sequence $|\mathscr{B}(X)| \to |\mathscr{B}(Y)| \to |\mathscr{B}(X)|//|\mathscr{B}(Y)|$. Since all objects in sight are cofibrant and the maps $|\mathscr{B}(X)| \to X$ and $|\mathscr{B}(Y)| \to Y$ are weak equivalences, the map induced on cofibers $|\mathscr{B}(X)|//|\mathscr{B}(Y)| \to Y//X$ is also a weak equivalence. So it suffices to show that the sequence

$$|\mathscr{B}(X)| \to |\mathscr{B}(Y)| \to |\mathscr{B}(X)|//|\mathscr{B}(Y)|$$

is a $(2n+k)$-homotopy fiber sequence. Geometric realization of simplicial T-algebras commutes with taking quotients, so all three T-algebras are realizations of simplicial objects. In a fixed simplicial dimension, all objects are freely generated by certain simplicial sets, and all maps are free maps. So by the previous paragraph, the sequence of bar resolutions is a $(2n+k)$-homotopy fiber sequence in every simplicial degree. But geometric realization preserves connectivity and homotopy fiber sequences of connected objects [9, Theorem B.4], which finishes the proof. $\square$

4. Stable homotopy

The goal of this section is to show that the stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over a certain Gamma-ring T^s. The theorem we are heading for has a well known algebraic analog which we first recall. The notion of an abelian group object in the category of T-algebras coincides with that of a model of T in the category $\mathscr{Ab}$ of simplicial abelian groups. The forgetful functor from abelian group objects to T-alg has a left adjoint and the category $\mathscr{Ab}(T)$ of abelian group objects is in fact an abelian category. Moreover, the category $\mathscr{Ab}(T)$ is equivalent to the category of modules over a certain simplicial ring T^{ab}, namely the endomorphism ring of the free abelian group object on one generator:

$$\mathscr{Ab}(T) \cong T^{\mathrm{ab}}\text{-mod}.$$

In Theorem 4.4 we are proving the homotopy theoretic analog of this fact. The idea is to replace abelian group objects by its homotopy analogue, namely infinite loop objects or spectra. Spectra of T-algebras form a model category $\mathscr{Sp}(T)$. Then there is another "ring" T^s whose modules are the spectra of T-algebras. We only have to allow Gamma-rings instead of simplicial rings, and instead of an equivalence of categories we obtain a Quillen equivalence of model categories

$$\mathscr{Sp}(T)_{\mathrm{conn}} \simeq T^s\text{-mod}.$$

T^s is the endomorphism Gamma-ring of the free T-algebra on one generator (see Section 4.6 for the precise meaning). The homotopy groups of T^s are isomorphic to the ring of stable homotopy operations of T-algebras. The two rings arising from the theory T are closely related. There is a 1-connected map of Gamma-rings $T^s \to HT^{\mathrm{ab}}$ which governs the relationship between stable homotopy and homology of T-algebras. The simplicial ring T^{ab} can be obtained from the Gamma-ring T^s by applying the functor L left adjoint to the Eilenberg–MacLane functor (see Theorem 5.2).

4.1. Spectra of T-algebras. To define spectra, we need to recall the definition of the suspension of a T-algebra X. Since T-algebras form a simplicial model category, the product $X \otimes_T S^1$ with the simplicial circle is defined. The suspension of X is then obtained as the cofiber, in the category of T-algebras, of the map $X \to X \otimes_T S^1$ induced by the inclusion of the unique vertex into S^1. ΣX is a bar construction with respect to the coproduct of T-algebras. This means that it is the geometric realization of the simplicial T-algebra which in simplicial degree k consists of the coproduct of k copies of X. The suspension functor has an adjoint Ω which is defined dually. The loop functor commutes with the forgetful functor, i.e., the underlying simplicial set of ΩX is the simplicial set of pointed maps of S^1 into the underlying simplicial set of X. Σ and Ω are a Quillen adjoint functor pair. The total derived functors of Σ and Ω are the suspension and loop functors on the homotopy category of T-algebras [32, I.2]. The simplicial structure thus provides liftings of these functors from the homotopy category to functors on the actual model category. One has to remember that ΣX can have the "wrong" homotopy type if X is not cofibrant just as ΩY can have the "wrong" homotopy type if Y is not fibrant.

For our purposes, the naive definition of a spectrum suffices. The following is an elaboration on the construction of [9, Section 2].

4.2. Definition. A spectrum X of T-algebras is a collection of T-algebras X_n, $n \geqslant 0$, and T-algebra homomorphisms $\Sigma X_n \to X_{n+1}$. A morphism of spectra $f: X \to Y$ is a collection of maps $f_n: X_n \to Y_n$ such that all the diagrams

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d] & \Sigma Y_n \arrow[d] \\
X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1}
\end{tikzcd}$$

commute. We denote the category of spectra by $\mathscr{S}\!\mathscr{p}(T)$.

If T is the theory of sets, a T-algebra is just a simplicial set and the above definition reduces to the notion of a spectrum as in [9, Section 2]. In general, a spectrum of T-algebras is the same thing as a model of T in the category of spectra of [9]. We call a map of spectra of T-algebras a weak equivalence (resp. fibration) if it is a stable weak equivalence (resp. stable fibration) as a map of spectra of simplicial sets. We call a map a cofibration if it has the left lifting property with respect to all acyclic fibrations of spectra of T-algebras.

4.3. Theorem. *If T is a simplicial theory, then the category $\mathscr{S}\!\mathit{p}(T)$ of spectra of T-algebras is a closed simplicial model category.*

Proof. We apply Lemma A.2 to lift the stable model category structure of spectra of simplicial sets to the category $\mathscr{S}\!\mathit{p}(T)$. The adjoint functor pair to use consists of the forgetful and the free T-algebra functor, applied dimensionwise to spectra. All limits, colimits as well as tensors and cotensor with simplicial sets are inherited from the category of T-algebras and they are defined dimensionwise for spectra of T-algebras. The category $\mathscr{S}\!\mathit{p}(T)$ is locally finitely presentable (Lemma A.1) and the model category of spectra of simplicial sets is cofibrantly generated (Lemma A.3). So it remains to describe a functor Q that provides fibrant replacements. In [9, Section 2] Bousfield and Friedlander use a functor Q given by

$$(QX)_n = \mathrm{colim}_i\ \Omega^i\, \mathrm{Sing}\,|X_{n+i}|.$$

(Kan's functor Ex^∞ can be substituted for the geometric realization of the singular complex). A priori, the functor Q is defined for spectra of simplicial sets; but this choice of Q is simplicial and preserves finite products, so it passes to the category of spectra of T-algebras. For every spectrum X, QX is an Ω-spectrum and degreewise a fibrant simplicial set. It is thus fibrant in the stable model category of spectra by [9, A.7]. So the lifting Lemma A.2 applies. □

Now we can state the main theorem of this section, saying that the (connective) stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over a certain Gamma-ring T^s. The fact that we only get connective spectra of T-algebras stems from the fact that Γ-spaces only represent connective spectra.

4.4. Theorem. *To a simplicial theory T there is functorially associated a Gamma-ring T^s. The ring $\pi_* T^s$ is isomorphic to the ring of stable homotopy operations of T-algebras. There is a Quillen adjoint functor pair*

$$T^s\text{-mod} \rightleftarrows \mathscr{S}\!\mathit{p}(T)$$

whose total derived functors are inverse equivalences between the homotopy category of T^s-modules and the homotopy category of connective spectra of T-algebras,

$$\mathrm{Ho}(T^s\text{-mod}) \;\cong\; \mathrm{Ho}(\mathscr{S}\!\mathit{p}(T))_{\mathrm{conn}}.$$

4.5. The Gamma-ring T^s. The Γ-space underlying the Gamma-ring T^s is defined as the composite of the free T-algebra functor, restricted to the category Γ^{op}, with the forgetful functor from T-algebras to pointed simplicial sets. The composite of the free T-algebra functor with the forgetful functor is a triple on the category of simplicial sets. As we pointed out in Section 2.3, this means that T^s comes with associative and unital maps

$$T^s \circ T^s \to T^s \quad \text{and} \quad \mathbb{S} \to T^s$$

making it a monoid in the category of Γ-spaces with respect to the composition product $\circ$. Composition with the assembly map (see Section 1.8)

$$T^s \wedge T^s \to T^s \circ T^s \to T^s$$

gives T^s a multiplication with respect to the smash product. Since the assembly map is associative and unital, T^s becomes a Gamma-ring.

4.6. A generalization: endomorphism Gamma-rings. The construction of the Gamma-ring T^s is a special case of a more general construction of endomorphism Gamma-rings. As input we can use any pointed category $\mathscr{C}$ which has finite coproducts. Then $\mathscr{C}$ is tensored over the category Γ^{op}, i.e., the assignment

$$X \wedge n^+ = \underbrace{X \amalg \ldots \amalg X}_{n}$$

is the object function of a functor $\wedge : \mathscr{C} \times \Gamma^{\mathrm{op}} \to \mathscr{C}$. As a consequence, the category $\mathscr{C}$ is also enriched over the category of Γ-sets. This means that for any two objects X and Y of $\mathscr{C}$ there is a homomorphism Γ-set $\mathrm{HOM}(X,Y)$ defined by

$$\mathrm{HOM}(X,Y)(n^+) = \mathscr{C}(X, Y \wedge n^+).$$

Furthermore there is a unit morphism $\mathbb{S} \to \mathrm{HOM}(X,X)$ induced by the identity of X and associative and unital composition pairings

$$\mathrm{HOM}(Y,Z) \wedge \ \mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Z).$$

The composition pairing is induced by the universal property of the smash product of Γ-sets from the maps

$$\mathscr{C}(Y, Z \wedge n^+) \wedge \ \mathscr{C}(X, Y \wedge m^+) \to \mathscr{C}(X, Z \wedge n^+ \wedge m^+); \quad f \wedge g \mapsto (f \wedge \mathrm{id}_{m^+}) \circ g.$$

In particular, for every object X, the endomorphism Γ-set $\mathrm{HOM}(X,X)$ becomes a Gamma-ring. If the category $\mathscr{C}$ is also enriched over the category of simplicial sets (as is the case for algebras over a simplicial theory), then the enrichment over Γ-sets extends to one over Γ-spaces. This basic observation gives a rich supply of (endomorphism) Gamma-rings. For a simplicial theory T the category of T-algebras is pointed, simplicially enriched and has coproducts. The value at n^+ of the endomorphism Gamma-ring of the free T-algebra on one generator is given by

$$\mathrm{HOM}(F^T(1^+), F^T(1^+))(n^+) \cong \mathrm{Hom}_{T\text{-alg}}(F^T(1^+), F^T(n^+))$$

which is naturally isomorphic to the value of the Gamma-ring T^s at n^+. Since the isomorphism also preserves the multiplications we obtain the following:

4.7. Lemma. *Let T be a simplicial theory. Then the Gamma-ring T^s is isomorphic to the endomorphism Gamma-ring of the free T-algebra on one generator.*

Warning. The homomorphism Γ-spaces $\mathrm{HOM}(X,Y)$ just defined are usually *not* adjoint to any kind of smash product pairing between $\mathscr{C}$ and the category of Γ-spaces. For example $\mathrm{HOM}(X,Y)$ usually does *not* preserve limits in the second variable (although it does take colimits in the first variable to limits). If $\mathscr{C} = \mathscr{GS}$ is the category of Γ-spaces, then the homomorphism Γ-space

$\mathrm{HOM}(X,Y)$ is different from the internal hom Γ-space $\mathrm{Hom}(X,Y)$ which is adjoint to the smash product. Indeed the former is made up from homomorphisms into Γ-spaces of the form $Y \wedge n^+$, whereas the latter uses maps into Γ-spaces of the form $Y_{n^+ \wedge -}$. The natural maps of Γ-spaces $Y \wedge n^+ \to Y_{n^+ \wedge -}$ induce a natural map $\mathrm{HOM}(X,Y) \to \mathrm{Hom}(X,Y)$.

4.8. Comparison of the stable categories. We now proceed to compare the two categories $\mathcal{S}\!p(T)$ and T^s-mod. This will be done through an intermediate category $\mathcal{GS}(T)$, the category of pointed functors $\Gamma^{\mathrm{op}} \to T$-alg, alias the models of T in the category of Γ-spaces. We obtain three model categories with Quillen adjoint functor pairs

$$T^s\text{-mod} \underset{\Psi_*}{\overset{\Psi_*}{\rightleftarrows}} \mathcal{GS}(T) \underset{\Phi(S,-)}{\overset{(-)(S)}{\rightleftarrows}} \mathcal{S}\!p(T).$$

The left pair is a Quillen equivalence, the right is a Quillen pair which passes to an equivalence of the homotopy category of $\mathcal{GS}(T)$ with that of connective spectra of T-algebras.

Step 1: We first establish the stable model category structure for $\mathcal{GS}(T)$. We call a map in $\mathcal{GS}(T)$ a weak equivalence (resp. fibration) if and only if it is a stable equivalence (resp. stable Q-fibration) of underlying Γ-spaces. A map is called a cofibration if it has the left lifting property with respect to all acyclic fibrations.

4.9. Theorem. *With these notions of fibrations, cofibrations and weak equivalences, the category $\mathcal{GS}(T)$ becomes a closed simplicial model category. If $X \to Y$ is a cofibration in $\mathcal{GS}(T)$, then for every pointed simplicial set K, the map $X(K) \to Y(K)$ is a cofibration of T-algebras.*

Proof. We want to apply Lemma A.2 to lift the stable Q-model category structure from Γ-spaces to $\mathcal{GS}(T)$. The adjoint functor pair to use consists of the forgetful and the free T-algebra functor, applied objectwise to Γ-objects. As a functor category into a complete and cocomplete simplicially enriched category, $\mathcal{GS}(T)$ has all limits and colimits as well as tensors and cotensor with simplicial sets. The category $\mathcal{GS}(T)$ is locally finitely presentable, see Lemma A.1. The crucial ingredient is the stably fibrant replacement functor Q. One possible choice of such Q is given by

$$(QX)(n^+) = \mathrm{colim}_i \Omega^i \,\mathrm{Sing}\,|X(S^i \wedge n^+)|.$$

Again we can use Ex^∞ [21] instead of geometric realization and singular complex. A priori, the functor Q is only defined on the category of Γ-spaces. However, Q is a simplicial functor and it preserves finite products, so it passes to an endofunctor on the category $\mathcal{GS}(T)$. There is a natural stable equivalence $X \to QX$, and QX is pointwise fibrant and very special, so it is fibrant in the stable Q-model category structure. Thus the lifting Lemma A.2 applies.

To get the statement about cofibrations we first consider generating cofibrations. These are of the form $X = T^s \circ A \to T^s \circ B = Y$ for $A \to B$ a cofibration of Γ-spaces. Then the map $X(K) \to Y(K)$ is obtained from a cofibration of simplicial sets by application of the free T-algebra functor, so it is a cofibration of T-algebras. The general case follows by the small object argument [37, Lemma A.1] since the property in question is preserved under cobase change, transfinite composition and retract. □

Step 2: The comparison of the category $\mathcal{S}\!p(T)$ of spectra of T-algebras with the category $\mathcal{GS}(T)$ of Γ-objects of T-algebras follows easily from the work of Bousfield and Friedlander. In [9, Section 5], they show that the functor $X \mapsto X(S)$ from Γ-spaces to spectra has a right adjoint $\Phi(S, -)$. Both functors are simplicial and the left adjoint preserves finite products, so they pass to adjoint functors between the categories $\mathcal{GS}(T)$ and $\mathcal{S}\!p(T)$ (see Lemma 2.5). Since (stable) weak equivalences and fibrations in $\mathcal{GS}(T)$ and $\mathcal{S}\!p(T)$ are defined on underlying Γ-spaces or spectra respectively, the functors $(-)(S)$ and $\Phi(S, -)$ still form a Quillen pair. We have to note here that the stable Q-model category structure for Γ-spaces has more fibrations than the stable Bousfield–Friedlander model category structure. If X is a connective fibrant spectrum of T-algebras, then the adjunction map $\Phi(S, X)(S) \xrightarrow{\sim} X$ is a stable equivalence. So if A is a cofibrant object in $\mathcal{GS}(T)$, then a map $A(S) \to X$ is a stable equivalence if and only if the adjoint map $A \to \Phi(S, X)$ is. Thus the Quillen adjoint functor pair passes to an equivalence between the homotopy category of $\mathcal{GS}(T)$ and the homotopy category of connective spectra of T-algebras.

Step 3: In this last step we want to construct a Quillen equivalence between the category $\mathcal{GS}(T)$ and the category of T^s-modules. The associative and unital assembly map $X \wedge Y \to X \circ Y$ of Section 1.8 gives a morphism

$$\Psi : T^s \wedge - \to T^s \circ -$$

of triples on the category of Γ-spaces. An algebra over the triple $T^s \wedge -$ is nothing but a T^s-module, an algebra over the triple $(T^s \circ -)$ is a Γ-object of T-algebras. Pulling back along the triple morphism Ψ gives a functor $\Psi^* : \mathcal{GS}(T) \to T^s\text{-mod}$. This functor has a left adjoint Ψ_* (see [24, Corollary 1]). The right adjoint Ψ^* preserves fibrations and weak equivalences since these are defined everywhere on underlying Γ-spaces; so the functors form a Quillen pair. Since the right adjoint in fact detects and preserves all weak equivalences, it suffices to show

4.10. Lemma. *For a cofibrant T^s-module A, the unit map $A \to \Psi^*\Psi_* A$ of the adjunction is a stable equivalence.*

Proof. We assume first that the cofibrant T^s-module A is induced, i.e., it is of the form $A = T^s \wedge Y$ for some cofibrant Γ-space Y. Pushforward along a map of triples takes free objects to free objects, so in this case the map in question is the assembly map

$$A = T^s \wedge Y \to T^s \circ Y = \Psi^*\Psi_* A,$$

which is a weak equivalence by [25, Proposition 5.23]. Now we assume that the cofibrant T^s-module A can be written as the pushout of a diagram of T^s-modules

$$A' \leftarrow K \rightarrowtail L$$

in which $K \to L$ is a cofibration and such that Lemma 4.10 holds for the cofibrant T^s-module A' and the quotient module $L/K \cong A/A'$. We claim that then the lemma also holds for A. Indeed, cofibrations of T^s-modules are injective and cofibers of T^s-modules are calculated on underlying Γ-spaces [37, Theorem 2.2]. So the cofiber sequence of T^s-modules $A' \to A \to A/A'$ gives rise to a long exact sequence of homotopy groups by [37, Lemma 1.3]. As a left adjoint in a Quillen functor pair Ψ_* preserves pushout, cofibers and cofibrations. Hence the five lemma gives the desired conclusion once we know that the cofiber sequence in $\mathcal{GS}(T)$

$$\Psi_* A' \to \Psi_* A \to \Psi_*(A/A')$$

also gives rise to a long exact sequence of homotopy groups. If we evaluate at the simplicial n-sphere S^n, we obtain a cofiber sequence of $(n-1)$-connected cofibrant T-algebras (by Theorem 4.9). By Theorem 3.6, we obtain a long exact sequence of homotopy groups in a stable range, and we let n go to infinity.

The previous two paragraphs together show that the conclusion of Lemma 4.10 holds for all T^s-modules which can be obtained from the trivial module by finitely many cobase changes along cofibrations between modules that are induced from Γ-spaces. Then it also holds for modules which are filtered direct limits of such modules. But an arbitrary cofibrant T^s-module is a retract of one of this sort by the small object argument [37, Lemma A.1]. □

4.11. Stable homotopy operations. Interpreting $\pi_* T^s$ in terms of stable homotopy operations is a standard representability argument. We will be brief since this result will not be used in the rest of this paper. A homotopy operation of T-algebras is a natural transformation $\pi_n \to \pi_m$ of functors T-alg $\to$ (sets) for some n, m. Homotopy operations can be composed if the source of one is the target of the other, and they form a category with objects the natural numbers. The functor π_n is represented by $F^T(S^n)$ in the homotopy category of T-algebras, i.e., $\pi_n X \cong [F^T(S^n), X]$ (the right-hand side denotes maps in the homotopy category). Consequently, the category of homotopy operations is isomorphic the opposite of the full subcategory of Ho (T-alg) generated by the $F^T(S^n)$. In particular, homotopy operations $\pi_n \to \pi_m$ are in bijective correspondence with elements of $[F^T(S^m), F^T(S^n)]$. Homotopy operations can be suspended, i.e., if $\tau : \pi_n \to \pi_m$ is one, one defines $\Sigma\tau : \pi_{n+1} \to \pi_{m+1}$ on a T-algebra X as the composite

$$\pi_{n+1}|X| \cong \pi_n \quad \Omega|X| \xrightarrow{\tau_{\Omega|X|}} \pi_m \quad \Omega|X| \cong \pi_{m+1}|X|.$$

The suspension of operations corresponds to the suspension

$$\Sigma : [F^T(S^m), F^T(S^n)] \to [F^T(S^{m+1}), F^T(S^{n+1})]$$

in the homotopy category of T-algebras.

A stable homotopy operation of degree n is represented by a sequence $(\tau_i)_{i \geqslant i_0}$ of homotopy operations $\tau_i : \pi_i \to \pi_{i+n}$ with the property that $\tau_{i+1} = \Sigma\tau_i$. Two such sequences define the same stable operation if the components eventually agree, i.e., if almost all components are equal. Stable homotopy operations can always be composed, the degrees add under composition, and they form a graded ring. The natural isomorphisms of $\pi_{n+i}|F^T(S^i)|$ with the sets of homotopy classes $[F^T(S^{n+i}), F^T(S^i)]$ assemble into an isomorphism of $\pi_n T^s = \mathrm{colim}_i\ \pi_{n+i}|F^T(S^i)|$ with the colimit of the sets $[F^T(S^{n+i}), F^T(S^i)]$ over suspension; but the elements of $\mathrm{colim}_i[F^T(S^{n+i}), F^T(S^i)]$ are nothing but the stable homotopy operations of degree n. The fact that the isomorphism between $\pi_* T^s$ and stable homotopy operations is multiplicative follows from the fact both products are (suitable kinds of) composition products, see Lemma 4.7.

5. Stable homotopy versus homology

We have seen that a simplicial theory T gives rise to two "rings" and a multiplicative map between them. There is the simplicial ring T^{ab} whose modules are the abelian group objects in the category of T-algebras. And there is the Gamma-ring T^s whose modules are (Quillen equivalent to) connective spectra of T-algebras. Abelianization induces a Gamma-ring map $T^s \to HT^{\mathrm{ab}}$. This map encodes the relationship between stable homotopy and homology in the homotopy theory of T-algebras. In this section we will show that the map $T^s \to HT^{\mathrm{ab}}$ is 1-connected and we will establish Hurewicz and Whitehead Theorems for T-algebras as well as universal coefficient and Atiyah–Hirzebruch spectral sequences.

5.1. Quillen homology. We recall Quillen's definition of homology as the left derived functor of abelianization [32, II.5]. By definition, the abelianization functor

$$-_{\mathrm{ab}}: T\text{-alg} \to \mathscr{Ab}(T)$$

is the left adjoint to the forgetful functor. If X is a T-algebra, one chooses a cofibrant replacement $X^{\mathrm{c}} \xrightarrow{\sim} X$ and defines the homology of X to be the homotopy of the abelianization of the replacement:

$$\mathrm{H}_* X = \pi_*(X^{\mathrm{c}}_{\mathrm{ab}}).$$

The unit of the adjunction is a map of T-algebras $X \to X_{\mathrm{ab}}$ which we refer to as the *Hurewicz map*. More generally there is (co-)homology of a T-algebra with coefficients. If M is a *right* simplicial T^{ab}-module, then the homology of X with coefficients in M is defined as the homotopy of the tensor product

$$\mathrm{H}_*(X; M) = \pi_*(M \otimes_{T^{\mathrm{ab}}} X^{\mathrm{c}}_{\mathrm{ab}}).$$

If N is a *left* simplicial T^{ab}-module, then the cohomology groups of X with coefficients in N are defined as the homotopy classes of T^{ab}-module maps

$$\mathrm{H}^*(X; N) = [X^{\mathrm{c}}_{\mathrm{ab}}, \Sigma^* N]_{T^{\mathrm{ab}}\text{-mod}}.$$

Note that unless the simplicial ring T^{ab} is commutative, homology and cohomology need different kinds of coefficients. In the case of the theory of sets this notion of homology specializes to singular homology of simplicial sets. For commutative rings it specializes to André–Quillen homology [31, Section 4].

Given a T-algebra X, we need a model for its suspension spectrum as a T^s-module. There is a slick definition using the interpretation of T^s as the endomorphism Gamma-ring of the free T-algebra on one generator as in Lemma 4.7. In the notation of Section 4.6 (with $\mathscr{C} = T$-alg), we can set $\Sigma^\infty X = \mathrm{HOM}(F^T(1^+), X)$ as a Γ-space, and with T^s-module structure given by the composition action of $T^s \cong \mathrm{HOM}(F^T(1^+), F^T(1^+))$. An equivalent description of $\Sigma^\infty X$ is as follows. First define a Γ-object of T-algebras $\widetilde{\Sigma^\infty} X$ by

$$(\widetilde{\Sigma^\infty} X)(n^+) \quad = \quad X \wedge n^+ \quad = \underbrace{X \amalg \cdots \amalg X}_{n},$$

the coproduct being taken in the category of T-algebras. As a functor $T\text{-alg} \to \mathcal{GS}(T)$, $\widetilde{\Sigma^\infty} X$ is left adjoint to evaluation at 1^+. Note that when $\widetilde{\Sigma^\infty} X$ is extended (by direct limit and degreewise application) to a functor from simplicial sets to T-algebras, then we get $(\widetilde{\Sigma^\infty} X\,)(K) = X \wedge K$ (this smash product refers to the enrichment of the category of T-algebras over pointed simplicial sets). Hence the spectrum associated to $\widetilde{\Sigma^\infty} X$ is isomorphic to the suspension spectrum of X as a T-algebra, which justifies the name. Recall from Section 4.8 that Γ-objects in T-alg can be pulled back to T^s-modules via a functor Ψ^*. This means that the underlying Γ-space of the Γ-T-algebra $\widetilde{\Sigma^\infty} X$ is endowed with a left T^s-action via the assembly map (1.8)

$$T^s \wedge \widetilde{\Sigma^\infty} X \quad \to \quad T^s \circ \widetilde{\Sigma^\infty} X \quad \to \quad \Sigma^\infty X.$$

The T^s-module $\Psi^* \widetilde{\Sigma^\infty} X$ is isomorphic to $\Sigma^\infty X$. So the two suspension spectrum objects $\Sigma^\infty X$ and $\widetilde{\Sigma^\infty} X$ have the same underlying Γ-spaces, but one is viewed as a T^s-module, the other one as an object of $\mathcal{GS}(T)$.

Recall from [37, Lemma 1.2] that the left adjoint functor L to the Eilenberg–MacLane functor H is strong symmetric monoidal and preserves finite products. In particular, it takes Gamma-rings to simplicial rings. Furthermore, the functors L and H pass to an adjoint functor pair between the category $\mathcal{GS}(T)$ of Γ-objects of T-algebras and the category $\mathcal{Ab}(T)$ of abelian group objects of T-algebras by Lemma 2.5. In the following theorem we combine these formal properties with some homotopical input to obtain information on the relationship between stable homotopy and homology for T-algebras.

5.2. Theorem. *For a T-algebra X, the object $L(\widetilde{\Sigma^\infty} X) \in \mathcal{Ab}(T)$ is naturally isomorphic to X_{ab}. The map of Gamma-rings $T^s \to HT^{\mathrm{ab}}$ induces an isomorphism on π_0 and an epimorphism on π_1 and its adjoint is an isomorphism of simplicial rings $L(T^s) \cong T^{\mathrm{ab}}$. In particular, if T is a discrete theory, then $T^{\mathrm{ab}} \cong \pi_0 T^s$. For a right T^{ab}-module M and a* T-algebra X, *there is a natural map of* Γ-*spaces*

$$HM \wedge^L_{T^s} \Sigma^\infty X \to H(M \otimes_{T^{\mathrm{ab}}} X_{\mathrm{ab}})$$

which is a stable equivalence whenever X is cofibrant.

Proof. The forgetful functor from abelian group objects factors as a composite

$$\mathcal{Ab}(T) \xrightarrow{H} \mathcal{GS}(T) \xrightarrow{\text{eval. at } 1^+} T\text{-alg}.$$

Since $L: \mathcal{GS}(T) \to \mathcal{Ab}(T)$ is left adjoint to H and $\widetilde{\Sigma^\infty}$ is left adjoint to evaluation at 1^+, their composite is a left adjoint to the forgetful functor. The adjoint of the map $T^s \to HT^{\mathrm{ab}}$ is a homomorphism of simplicial rings so it suffices to show that it is an isomorphism in $\mathcal{Ab}(T)$. But the Γ-space T^s underlies the suspension spectrum of the free T-algebra on one generator, so $L(T^s)$ is isomorphic to the free abelian group object on one generator T^{ab} by what we already proved. The fact that $T^s \to HT^{\mathrm{ab}}$ induces an isomorphism on π_0 and an epimorphism on π_1 then follows from [37, Lemma 1.2].

There is a functorial cofibrant replacement $(\Sigma^\infty X)^c \to \Sigma^\infty X$ in the category of T^s-modules. Then [37, Lemma 4.2] with $R = T^s$ provides a natural stable equivalence of Γ-spaces from $HM \wedge_{T^s} (\Sigma^\infty X)^c$ to $H(M \otimes_{T^{\mathrm{ab}}} L((\Sigma^\infty X)^c))$. Since $X_{\mathrm{ab}} \cong L(\widetilde{\Sigma^\infty} X)$, it remains to show that the map of cofibrant T^{ab}-modules $L((\Sigma^\infty X)^c) \to L(\widetilde{\Sigma^\infty} X)$ is a weak equivalence. This follows if we can show that for any T^{ab}-module W, the induced map on homomorphism spaces

$$\hom_{T^{\mathrm{ab}}\text{-mod}}(L(\widetilde{\Sigma^\infty} X), W) \to \hom_{T^{\mathrm{ab}}\text{-mod}}(L((\Sigma^\infty X)^c), W)$$

is a weak equivalence. By the various adjunctions, this map is isomorphic to the map

$$\hom_{\mathscr{GS}(T)}(\widetilde{\Sigma^\infty} X, HW) \to \hom_{\mathscr{GS}(T)}(\Psi_*(\Sigma^\infty X)^c, HW) \cong \hom_{T^s\text{-mod}}((\Sigma^\infty X)^c, HW)$$

induced by the stable equivalence of cofibrant objects $\Psi_*(\Sigma^\infty X)^c \to \widetilde{\Sigma^\infty} X$ (this uses Lemma 4.10). Hence the latter map of homomorphism spaces is a weak equivalence, which finishes the proof. □

5.3. Corollary (Hurewicz theorem). *Let T be a simplicial theory and X a cofibrant $(n-1)$-connected T-algebra $(n \geqslant 2)$. Then the Quillen homology of X vanishes below dimension n and the Hurewicz map induces an isomorphism $\pi_n X \cong \mathrm{H}_n X$ and an epimorphism $\pi_{n+1} X \to \mathrm{H}_{n+1} X$.*

Proof. An application of Theorem 3.6 to the cofiber sequence $X \to \mathrm{Cone}(X) \to \Sigma X$ shows that the map $|X| \to \Omega|\Sigma X|$ is $(2n-1)$-connected. Since $n \geqslant 2$, the nth homotopy group of X is thus isomorphic to the nth stable homotopy group and $\pi_{n+1} X$ surjects onto the $(n+1)$st stable homotopy group. By Theorem 5.2, the Γ-space HX_{ab} is stably equivalent to $HT^{\mathrm{ab}} \wedge^L_{T^s} \Sigma^\infty X$. So the Tor spectral sequence for the derived smash product [37, Lemma 3.1] takes the form

$$\mathrm{Tor}_p^{\pi_* T^s}(\pi_* T^{\mathrm{ab}}, \pi_* \Sigma^\infty X)_q \quad \Rightarrow \quad \mathrm{H}_{p+q}(X).$$

Since the map $T^s \to HT^{\mathrm{ab}}$ is 1-connected, this spectral sequence gives the desired answer for the first non-trivial homology groups of X. □

5.4. Corollary (Whitehead theorem). *Let T be a simplicial theory and $X \to Y$ a map of simply connected T-algebras which induces an isomorphism in Quillen homology. Then the map is a weak equivalence.*

Proof. We can assume that X and Y are 1-reduced and cofibrant and that the map is a cofibration. Then the cofiber $Y//X$ is 1-connected and has vanishing Quillen homology, hence is weakly contractible by the Hurewicz Theorem 5.3. Again by the Hurewicz Theorem, the map $X \to Y$ is 2-connected. An application of Theorem 3.6 to the cofibration $X \to Y$ and induction gives that the map $X \to Y$ is m-connected for all m. □

5.5. Spectral sequences relating Quillen homology and stable homotopy. Let X be a cofibrant T-algebra and M a coefficient module for Quillen homology. By Theorem 5.2, the homology groups of X with coefficients in M are isomorphic to the stable homotopy groups of the Γ-space

$HM \wedge^L_{T^s} \Sigma^\infty X$. So the spectral sequence for the derived smash product [37, Lemma 3.1] gives a *universal coefficient spectral sequence*

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{\pi_* T^s}_p(\pi_* M, \pi^s_* X)_q \quad \Rightarrow \quad \mathrm{H}_{p+q}(X; M).$$

This spectral sequence lies in the first quadrant. If X is $(n-1)$-connected, if we take $M = T^{\mathrm{ab}}$ and if for simplicity we take T to be a discrete theory we can read off the following six term exact sequence

$$(\pi_1 T^s \otimes \pi^s_{n+1} X) \oplus (\pi_2 T^s \otimes \pi^s_n X) \to \pi^s_{n+2} X \to \mathrm{H}_{n+2} X \to \pi_1 T^s \otimes \pi^s_n X \to \pi^s_{n+1} X \to \mathrm{H}_{n+1} X \to 0$$

which refines the part of the Hurewicz theorem that claims the surjectivity of the Hurewicz map in dimension $n+1$.

If W is a right T^s-module, then the homotopy groups of the derived smash product $W_* X = \pi_*(W \wedge^L_{T^s} \Sigma^\infty X)$ are a generalized homology theory in the T-algebra X. The spectral sequence [37, Lemma 3.1] together with Theorem 5.2 thus gives an *Atiyah–Hirzebruch spectral sequence*

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(X; \pi_q W) \Rightarrow W_{p+q}(X).$$

6. Relation to theory cohomology

In this section we provide the link between the cohomology and the stable homotopy of an algebraic theory. In [20, Definition 4.2], Jibladze and Pirashvili introduce the cohomology of an algebraic theory as Ext groups in an abelian functor category — see Remark 6.4 for some background and explanation about their cohomology theory. The main result of this section, Theorem 6.7, says that the Jibladze–Pirashvili homology groups of a theory T with coefficients in a functor G are isomorphic to the topological Hochschild homology groups of the Gamma-ring T^s with coefficients in a bimodule $G^!$ associated to G. This generalizes a theorem of Pirashvili and Waldhausen [30, Theorem 3.2]. We also show that the analogous statement in cohomology holds provided the coefficient functor G is additive.

6.1. Homological algebra in functor categories. Let $\mathscr{C}$ be a small category with zero object and R any ring. We denote by $\mathscr{F}(\mathscr{C}, R)$ the category of covariant pointed functors from $\mathscr{C}$ to the category of left R-modules. This is an abelian category in which exactness is defined objectwise. For every object c of $\mathscr{C}$ there are functors P_c and I_c defined by

$$P_c(d) = \tilde{R}[\mathscr{C}(c,d)] \quad \text{and} \quad I_c(d) = \mathrm{map}_*(\mathscr{C}(d,c), R_{\mathrm{inj}}).$$

Here $\tilde{R}[-]$ denotes the reduced free R-module on the pointed set of morphisms from c to d, $R_{\mathrm{inj}} = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is the injective cogenerator in the category of left R-modules and "map_*" denotes the set of pointed maps into R_{inj} with the pointwise left R-module structure. Because of the Yoneda-type isomorphisms

$$\mathrm{Hom}_{\mathscr{F}(\mathscr{C},R)}(P_c, G) \cong G(c) \quad \text{and} \quad \mathrm{Hom}_{\mathscr{F}(\mathscr{C},R)}(G, I_c) \cong \mathrm{Hom}_{R\text{-mod}}(G(c), R_{\mathrm{inj}}),$$

P_c is a projective object and I_c is an injective object in the abelian category $\mathscr{F}(\mathscr{C},R)$. Furthermore, the functors P_c form a set of projectives generators and the functors I_c form a set of injectives cogenerators for $\mathscr{F}(\mathscr{C},R)$ when c runs over the objects of $\mathscr{C}$.

6.2. (Co-)homology of algebraic theories. We consider a *discrete, pointed* algebraic theory T. We abbreviate to $\mathscr{F}(T)$ the abelian category $\mathscr{F}(T^{\mathrm{op}}, T^{\mathrm{ab}})$ of pointed functors from T^{op} to the category of left T^{ab}-modules. We recall from [6, Proposition 3.8.5] that T^{op} is equivalent to the full subcategory of T-alg given by the finitely generated free T-algebras. Also the category of left modules over the ring T^{ab} is equivalent to the category $\mathscr{Ab}(T)$ of abelian group objects of T-algebras. T^{ab} is the endomorphism ring of the free abelian group object on one generator, and by Theorem 5.2 it is isomorphic to the ring $\pi_0 T^s$. The category $\mathscr{F}(T)$ has a special object I_{ab}, the abelianization functor for T-algebras, restricted to T^{op}. Every $(R\otimes(T^{\mathrm{ab}})^{\mathrm{op}})$-module M defines a functor $M\otimes_{T^{\mathrm{ab}}}I_{\mathrm{ab}}$ in $\mathscr{F}(T^{\mathrm{op}},R)$. These are precisely the *additive* functors, i.e., those functors which commute with coproducts. The functor R-mod-$T^{\mathrm{ab}} \to \mathscr{F}(T^{\mathrm{op}},R)$ which sends M to $M\otimes_{T^{\mathrm{ab}}}I_{\mathrm{ab}}$ is right adjoint to the functor which sends $G\in\mathscr{F}(T^{\mathrm{op}},R)$ to the R-T^{ab}-bimodule

$$G^{\mathrm{add}} = \mathrm{coker}(G(2^+) \xrightarrow{(p_1)_*^+(p_2)_*-\nabla_*} G(1^+)).$$

Here the right T^{ab}-action is induced, through the functor G, from the action of the monoid $\mathrm{hom}_T(1^+,1^+)$ on the free T-algebra on one generator.

In the case where $R = T^{\mathrm{ab}}$ and $G\in\mathscr{F}(T)$, the abelian group G^{add} thus has a two-sided action of the ring T^{ab}. In this case we can equalize the actions and define

$$\mathscr{Q}G = G^{\mathrm{add}}/(tx - xt),$$

i.e., we divide out the subgroup generated by elements of the form $tx - xt$ for $x\in G^{\mathrm{add}}$ and $t\in T^{\mathrm{ab}}$. Then $\mathscr{Q}$ is an additive, right exact functor from $\mathscr{F}(T)$ to the category of abelian groups and so it has left derived functors $L_i\mathscr{Q}$.

6.3. Definition (Jibladze and Pirashvili [20, Definition 4.2] **).** Let T be a pointed discrete algebraic theory and $G\in\mathscr{F}(T)$. The homology and cohomology of T with coefficients in G are then defined as

$$\mathrm{H}_*(T;G) = (L_*\mathscr{Q})(G) \quad \text{and} \quad \mathrm{H}^*(T;G) = \mathrm{Ext}^*_{\mathscr{F}(T)}(I_{\mathrm{ab}}, G).$$

6.4. Remark. The notion of (co-)homology of a theory T with coefficients in a functor in the sense of Jibladze and Pirashvili has to be distinguished from the Quillen (co-)homology of a T-algebra X with coefficients in an abelian group object which we reviewed in Section 5.1. If the theory T is fixed, then Quillen homology provides a homology theory for varying T-algebras. For example, Quillen homology satisfies excision and is homotopy invariant.

The Jibladze–Pirashvili cohomology plays the same role for algebraic theories that is played by Hochschild cohomology for algebras over a field, and it generalizes MacLane cohomology [26] for arbitrary rings. For example in [20, Section 4], Jibladze and Pirashvili give interpretations of the theory cohomology groups in dimensions 0, 1 and 2 as suitable "center", "outer derivation" and "singular extension" groups respectively. For the theory of modules over a ring and for an additive coefficient functor these reduce to the corresponding classical interpretations of the MacLane cohomology groups. Indeed, by [20, Theorem A] the cohomology groups of the theory of R-modules with coefficients in a bimodule are isomorphic to the MacLane cohomology groups.

6.5. Topological Hochschild (co-)homology. Let S be a Gamma-ring and M an S-bimodule. We choose a cofibrant approximation $cS \to S$ in the model category of Gamma-rings of [37, Theorem 2.5].

For us the *topological Hochschild homology* groups of S with coefficients in M are defined as the homotopy groups of the derived smash product of S and M as cS-S-bimodules,

$$\mathrm{THH}_n(S; M) = \pi_n(S \wedge^L_{cS \wedge S^{\mathrm{op}}} M).$$

This is not the original definition of topological Hochschild homology given by Bökstedt [5]. However Shipley [39, Section 4] shows in the context of symmetric spectra that the two definitions are equivalent; a proof of the analogous statements in the context of Gamma-rings is similar, but easier. The *topological Hochschild cohomology* groups of S with coefficients in M are defined as the homotopy classes of cS-S-bimodule maps from S to M,

$$\mathrm{THH}^n(S; M) = \begin{cases} [S, \Sigma^n M]_{cS\text{-}S} & \text{if } n \geqslant 0, \\ [\Sigma^{-n} S, M]_{cS\text{-}S} & \text{if } n < 0. \end{cases}$$

Here Σ refers to the suspension functor in the homotopy category of cS-S-bimodules.

6.6. The bimodule construction. To a functor $G \in \mathscr{F}(T^{\mathrm{op}}, R)$ there is a functorially associated cHR-T^s-bimodule $G^!$. This generalizes the construction of [30, Example 2.6]. Here cHR is a cofibrant approximation of HR in the model category of Gamma-rings of [37, Theorem 2.5]. As a Γ-space, $G^!$ is equal to the composite functor

$$\Gamma^{\mathrm{op}} \xrightarrow{F^T} T^{\mathrm{op}} \xrightarrow{G} R\text{-mod} \xrightarrow{\text{forget}} (\text{pt. simpl. sets}).$$

In other words, the value of the Γ-space $G^!$ on n^+ is the underlying set of the value of G on the free T-algebra on n generators. There is a map $HR \circ G^! \circ T^s \to G^!$ given at $n^+ \in \Gamma^{\mathrm{op}}$ by

$$(HR \circ G^! \circ T^s)(n^+) = \tilde{R}[G(F^T(F^T(n^+)))] \longrightarrow G(F^T(n^+)) = G^!(n^+);$$

this map is evaluation both inside and outside of G and it uses that G takes values in R-modules and that F^T is a triple. Composition with the assembly map (1.8) and the stable equivalence of Gamma-rings $cHR \xrightarrow{\sim} HR$ gives the bimodule structure $cHR \wedge G^! \wedge T^s \to G^!$. For example, the cHR-T^s-bimodule associated to the additive functor $M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}$ is the Eilenberg–MacLane module HM.

6.7. Theorem. *Let T be a pointed discrete algebraic theory and $G \in \mathscr{F}(T)$. There is a natural isomorphism*

$$\mathrm{H}_*(T; G) \cong \mathrm{THH}_*(T^s; G^!).$$

For T^{ab}-bimodules M, the groups $\mathrm{THH}^*(T^s;HM)$ *are trivial in negative dimensions and for* $* \geqslant 0$ *there is a natural isomorphism*

$$\mathrm{H}^*(T;M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}) \cong \mathrm{THH}^*(T^s;HM).$$

6.8. Remark. A special case of interest is the case when T is the theory of left R-modules for a ring R. We are then looking at functors $G \in \mathscr{F}(R)$ from the category of finitely generated free R-modules to all R-modules. In this case the homotopy groups $\pi_* G^!$ are (essentially by definition) the stable derived functors of G in the sense of Dold and Puppe [13, Section 8.3]. The homological case of Theorem 6.7 then specializes to [30, Theorem 3.2]. By a theorem of Jibladze and Pirashvili [20, Theorem A] the groups $\mathrm{Ext}^*_{\mathscr{F}(R)}(I, M \otimes_R -)$ are naturally isomorphic to the MacLane cohomology groups $\mathrm{H}^*_{\mathrm{ML}}(R;M)$ introduced in [26]. So the cohomological part of Theorem 6.7 implies that MacLane cohomology coincides with topological Hochschild cohomology.

The cohomology of T with coefficients in a non-additive functor can differ from the topological Hochschild cohomology, see Remark 6.15. To prove the homological part of Theorem 6.7 we use the same strategy as [30]: we show that topological Hochschild homology has the universal properties of the derived functors of $\mathscr{Q}$. The cohomological part follows from a comparison of the derived category of the abelian category $\mathscr{F}(T^{\mathrm{op}},R)$ with the homotopy category of cHR-T^s-bimodules. We start with three short lemmas.

6.9. Lemma. *Let P_n be the projective object of $\mathscr{F}(T^{\mathrm{op}},R)$ represented by $n^+ \in T^{\mathrm{op}}$ (see 6.1). Then $P_n^!$ is stably equivalent to $cHR \wedge n^+ \wedge T^s$ as a cHR-T^s-bimodule.*

Proof. The Γ-space underlying $P_n^!$ is the composite of three other Γ-spaces, $P_n^! = HR \circ \Gamma^n \circ T^s$. The cHR-T^s-bimodule structure comes through the left and right composition factors. Since cHR is cofibrant as a Gamma-ring, it is also cofibrant as a Γ-space [37, Theorem 2.5]. By [25, Proposition 5.23] the assembly map

$$cHR \wedge \Gamma^n \wedge T^s \xrightarrow{\sim} HR \circ \Gamma^n \circ T^s$$

from the smash to the composition product is thus a stable equivalence. The lemma follows since the cofibrant Γ-spaces Γ^n and $\mathbb{S} \wedge n^+$ are stably equivalent. □

Recall from [37, Section 4] that the functor L which is adjoint to the Eilenberg–MacLane functor H passes to a functor $L\colon cHR\text{-mod-}T^s \to R\text{-mod-}T^{\mathrm{ab}}$ (using the isomorphism $T^{\mathrm{ab}} \cong L(T^s)$).

6.10. Lemma. *There is a natural isomorphism $G^{\mathrm{add}} \cong LG^!$ of functors $\mathscr{F}(T^{\mathrm{op}},R) \to R$-mod-$T^{\mathrm{ab}}$.*

Proof. The evaluation map $\tilde{\mathbb{Z}}[G(1^+)] \to G(1^+)$ passes to a natural map of R-T^{ab}-bimodules from $LG^!$ to G^{add}. By Lemma 6.9 and [37, Lemma 1.2], $LP_n^!$ is isomorphic to the free bimodule $(R \otimes T^{\mathrm{ab}})^n$, so the map is an isomorphism for the projective generators. Since both expressions are right exact in G, the map is an isomorphism in general. □

We denote by $s\mathscr{F}(T^{op},R)$ the category of simplicial objects in $\mathscr{F}(T^{op},R)$ (which is the same as the category of pointed functors from T^{op} to the category of simplicial left R-modules). The bimodule construction 6.6 which takes G to $G^!$ can be applied dimensionwise to simplicial functors.

6.11. Lemma. *The bimodule construction $G \mapsto G^!$ takes short exact sequences of simplicial functors in $s\mathscr{F}(T^{op},R)$ to homotopy cofiber sequences of cHR-T^s-bimodules.*

Proof. When the underlying Γ-spaces of the bimodules associated to a short exact sequence are evaluated at a simplicial sphere S^n, one obtains a short exact sequence of simplicial R-modules which give rise to a long exact sequence in homotopy. When n tends to infinity, these assemble into a long exact sequence for the homotopy groups of the cHR-T^s-bimodules. $\square$

Proof of the homological part of Theorem 6.7. We show that the functors $\mathrm{THH}_*(T^s;(-)^!)$ have the universal properties of the derived functors of $\mathscr{Q}$. By [37, Lemmas 1.2 and 4.1] we can identify $\mathrm{THH}_0(T^s;G^!)$ as

$$\mathrm{THH}_0(T^s;G^!) \cong L(T^s \wedge_{cT^s \wedge (T^s)^{op}} G^!) \cong T^{ab} \otimes_{T^{ab} \otimes (T^{ab})^{op}} LG^!.$$

So by Lemma 6.10, the group $\mathrm{THH}_0(T^s,G^!)$ is naturally isomorphic to $\mathscr{Q}G$. Short exact sequences of objects in $\mathscr{F}(T)$ go to homotopy cofiber sequences of bimodules (Lemma 6.11), which become homotopy cofiber sequences of Γ-spaces after derived smash product with T^s over $cT^s \wedge (T^s)^{op}$. So the functors $\mathrm{THH}_*(T^s,(-)^!)$ have a connecting homomorphism with respect to which short exact sequences of objects in $\mathscr{F}(T)$ go to long exact sequences in homology. So it remains to show that topological Hochschild homology vanishes in positive dimensions for each of the projective generators P_n of $\mathscr{F}(T)$. Using Lemma 6.9 we calculate

$$\mathrm{THH}_*(T^s,P_n^!) \cong \pi_*(T^s \wedge^L_{cT^s \wedge (T^s)^{op}} (HT^{ab} \wedge n^+ \wedge T^s)) \cong \pi_*(HT^{ab} \wedge n^+) \cong (\pi_* T^{ab})^n$$

which is indeed trivial in positive dimensions since T^{ab} is a discrete ring. $\square$

6.12. Model structures for simplicial functors. The cohomological part of Theorem 6.7 is a special case of a more general statement about the relationship between the category of simplicial functor from T^{op} to R-modules and the category of cHR-T^s-bimodules. Quillen [32, II.4, Theorem 4] provides a standard model category structure on the category $s\mathscr{F}(T^{op},R)$ of simplicial functors. The weak equivalences (resp. fibrations) are the maps which are objectwise weak equivalences (resp. fibrations) of simplicial R-modules. We refer to this model category structure as the *strict* structure for simplicial functors. By the Dold-Kan theorem the normalized chain complex functor induces an equivalence of the strict homotopy category of $s\mathscr{F}(T^{op},R)$ with the derived category $\mathscr{D}^+(\mathscr{F}(T^{op},R))$ of non-negative dimensional chain complexes over the abelian category $\mathscr{F}(T^{op},R)$. The bimodule construction takes objectwise weak equivalences of simplicial functors to stable equivalences of cHR-T^s-bimodules. Lemma 6.11 implies that the induced functor on the level of homotopy categories

$$(-)^! : \mathscr{D}^+(\mathscr{F}(T^{op},R)) \rightarrow \mathrm{Ho}(cHR\text{-mod-}T^s)$$

is a triangulated functor.

We call a map of simplicial functors $F \to G$ a stable equivalence (resp. stable fibration) if the associated map of cHR-T^s-bimodules $F^! \to G^!$ is a stable equivalence (resp. stable fibration). The stable cofibrations coincide with the strict cofibrations. A simplicial functor G is stably fibrant if and only if it is *homotopy-additive*, i.e., if for all $X, Y \in T^{\mathrm{op}}$ the map $F(X) \oplus F(Y) \to F(X \amalg Y)$ is a weak equivalence of simplicial R-modules.

6.13. Theorem. *The stable notions of fibrations, cofibrations and weak equivalences make the category $s\mathscr{F}(T^{\mathrm{op}}, R)$ of simplicial functors into a closed simplicial model category. The functor $(-)^!$ is the right adjoint of a Quillen equivalence between the stable model category of simplicial functors $s\mathscr{F}(T^{\mathrm{op}}, R)$ and the model category of cHR-T^s-bimodules.*

Proof. The functor $G \mapsto G^!$ preserves all limits and we first want to see that it actually has a left adjoint. It suffices to show this for discrete functors in $\mathscr{F}(T^{\mathrm{op}}, R)$. The category $\mathscr{F}(T^{\mathrm{op}}, R)$ is complete, it has a set of cogenerators (see Section 6.1) and it is well-powered (i.e., every object has only a set of subobjects). So Freyd's Special Adjoint Functor Theorem (see e.g. [27, Section V.8, Corollary]) provides a left adjoint $(-)_!$. To obtain the model category structure we apply the lifting Lemma A.2. The category $s\mathscr{F}(T^{\mathrm{op}}, R)$ of simplicial functors is complete, cocomplete, simplicially enriched and locally finitely presentable (Lemma A.1). The model category structure of cHR-T^s-bimodules is cofibrantly generated. It remains to find a stably fibrant replacement functor Q for the category $s\mathscr{F}(T^{\mathrm{op}}, R)$. We first note that a simplicial functor $G \in s\mathscr{F}(T^{\mathrm{op}}, R)$ can be extended to a functor from the category of T-algebras to simplicial R-modules by the coend construction

$$G(X) = \int^{k^+ \in T^{\mathrm{op}}} X^k \wedge G(k^+) \quad \text{for } X \in T\text{-alg}.$$

Then the functor Q is given by

$$(QG)(k^+) = \mathrm{colim}_n \quad \Omega^n G(\Sigma^n F^T(k^+)).$$

The underlying Γ-space of $(QG)^!$ is a stably fibrant replacement on the Γ-space underlying $G^!$, so Q in fact has the properties needed to apply the lifting Lemma A.2. We conclude that the stable notions of cofibrations, fibrations and weak equivalences make the category $s\mathscr{F}(T^{\mathrm{op}}, R)$ of simplicial functors into a closed simplicial model category.

By definition the right adjoint $(-)^!$ preserves and detects weak equivalences and fibrations. So it remains to show that for every cofibrant cHR-T^s-bimodule A the unit map $A \to (A_!)^!$ is a stable equivalence. This is very similar to Lemma 4.10. We first consider the case when A is one of the generating bimodules, i.e., when it is of the form $A = cHR \wedge (\Gamma^n \wedge K) \wedge T^s$ for some pointed simplicial set K. Then $A_! \cong P_n \otimes \tilde{\mathbb{Z}}[K]$ and the unit map

$$A = cHR \wedge (\Gamma^n \wedge K) \wedge T^s \to HR \circ (\Gamma^n \wedge K) \circ T^s = (A_!)^!$$

is the composite of the assembly map and the map induced by the stable equivalence $cHR \to HR$. The assembly map is a stable equivalence when all except possibly one of the factors are cofibrant [25, Proposition 5.23]. Since cHR is cofibrant as a Gamma-ring, it is also cofibrant as a Γ-space [37, Theorem 2.5]. Hence the map $A \to (A_!)^!$ is a stable equivalence if A is a generating bimodule.

An arbitrary cofibrant cHR-T^s-bimodule is obtained from the trivial bimodule by iterated pushouts along cofibrations between bimodule of the above form, transfinite composition and retract. So the rest of the argument is exactly as in Lemma 4.10. We only have to observe that the functor $(-)_!$ takes cofiber sequences of cHR-T^s-bimodules to cofiber sequences of simplicial functors. But cofiber sequences in $s\mathscr{F}(T^{\mathrm{op}}, R)$ are in particular short exact sequences which posses long exact sequences in homotopy by Lemma 6.11. $\square$

If we take $R = T^{\mathrm{ab}}$ and $F = I_{\mathrm{ab}}$ in the following corollary, the left hands side becomes $\mathrm{H}^*(T; M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}})$ and the right-hand side becomes $[HT^{\mathrm{ab}}, HM]^*_{cHT^{\mathrm{ab}}\text{-}T^s}$. Change of rings gives an isomorphism

$$[HT^{\mathrm{ab}}, HM]^*_{cHT^{\mathrm{ab}}\text{-}T^s} \cong [T^s, HM]^*_{cT^s\text{-}T^s} = \mathrm{THH}^*(T^s; HM),$$

so the cohomological part of Theorem 6.7 is a special case of

6.14. Corollary. *Let F be any functor in $\mathscr{F}(T^{\mathrm{op}}, R)$ and M an R-T^{ab}-bimodule. Then the groups* $[\Sigma^n F^!, HM]_{cHR\text{-}T^s}$ *are trivial for $n > 0$ and the functor $(-)^!$ induces natural isomorphisms*

$$\mathrm{Ext}^n_{\mathscr{F}(T^{\mathrm{op}}, R)}(F, M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}) \cong [F^!, \Sigma^n HM]_{cHR\text{-}T^s}.$$

Proof. For an arbitrary cHR-T^s-bimodule W, the group $[W, HM]_{cHR\text{-}T^s}$ is isomorphic to the group of R-T^{ab}-bimodule homomorphism from $\pi_0 W$ to M. Since $\pi_0(\Sigma^n F^!)$ is trivial for $n > 0$, the first claim follows. When we regard functors F and G in $\mathscr{F}(T^{\mathrm{op}}, R)$ as constant simplicial objects, $\mathrm{Ext}^n_{\mathscr{F}(T^{\mathrm{op}}, R)}(F, G)$ is isomorphic to the maps from F to $\Sigma^n G$ in the strict homotopy category of simplicial functors $s\mathscr{F}(T^{\mathrm{op}}, R)$. The functor $M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}$ is additive and the associated bimodule HM is stably fibrant, so $M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}$ is fibrant in the stable model category structure of simplicial functors. So the maps from F to $M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}}$ coincide in the strict and stable homotopy categories. Since $(-)^!$ is the right adjoint of a Quillen equivalence of model categories (Theorem 6.13), the group of maps from F to $\Sigma^n(M \otimes_{T^{\mathrm{ab}}} I_{\mathrm{ab}})$ in the stable homotopy category of simplicial functors is mapped isomorphically to the group of maps from $F^!$ to HM in $\mathrm{Ho}(cHR\text{-mod-}T^s)$. $\square$

6.15. Remark. The map $\mathrm{Ext}^n_{\mathscr{F}(T^{\mathrm{op}}, R)}(F, G) \to [F^!, \Sigma^n G^!]_{cHR\text{-}T^s}$ is not bijective for arbitrary functors in $\mathscr{F}(T^{\mathrm{op}}, R)$. For example, if we take F to be one of the projective generators P_n then $\mathrm{Hom}_{\mathscr{F}(T^{\mathrm{op}}, R)}(P_n, G) \cong G(n^+)$, but $[P_n^!, G^!]_{cHR-T^s} \cong (\pi_0 G^!)^n$ by Lemma 6.9. These two expressions are different unless G is additive. In particular we can take T to be the theory of pointed sets and $R = T^{\mathrm{ab}} = \mathbb{Z}$. In this case the abelianization functor I_{ab} is isomorphic to the projective object P_1, so $\mathrm{H}^0(T; G) \cong G(1^+)$ and $\mathrm{H}^*(T; G)$ is trivial for $* \geqslant 1$. On the other hand T^s is the sphere Gamma-ring, so $\mathrm{THH}^*(T^s; G^!) \cong \pi_* G^!$, which can have higher homotopy groups.

7. Examples

7.1. Sets. In the theory of pointed sets, the algebras are the pointed simplicial sets and the stable category is (a model for) the usual stable homotopy category. The associated Gamma-ring is the sphere spectrum $\mathbb{S}$, so Theorem 4.4 reduces to [9, Theorem 5.8] saying that the homotopy theory of Γ-spaces is equivalent to that of connective spectra.

7.2. Simplicial sets with G-action. Let G be a simplicial monoid and consider the theory of pointed simplicial sets with pointed G-action. The stable category is the category of spectra with G-action (i.e., G-objects in the category of spectra in the sense of [9]). The stable equivalences are equivariant maps which induce isomorphisms of the homotopy groups of underlying spectra. The associated Gamma-ring is $\mathbb{S}[G]$, the monoid ring of G over the sphere spectrum (see 1.11). The map from stable homotopy to homology is represented by the map of monoid rings $\mathbb{S}[G] \to H(\mathbb{Z}[G])$.

If G is a simplicial group (not just a simplicial monoid), then the homotopy theory of pointed G-simplicial sets is the same as the homotopy theory of retractive spaces over the classifying space BG. This is well known and can be seen as follows: we let EG denote a universal principal G-space, i.e., any weakly contractible simplicial set with a free G-action, and we take the orbit space of EG by the G-action as our model for the classifying space. Then pullback along the orbit map $EG \to BG$ is an equivalence of categories between the category of simplicial sets containing BG as a retract, and the category of (unpointed) G-simplicial sets containing EG as an equivariant retract (in both cases the section and retraction are part of the data).

The appropriate model category structure for retractive G-spaces over EG is the one in which fibrations and weak equivalences are those morphisms that are fibrations and weak equivalences of simplicial sets after forgetting the G-action, the retraction and the section. The functor that collapses the retract EG to a point is then the left adjoint of a Quillen equivalence between the category of retractive G-simplicial sets over EG, and the category of pointed G-simplicial sets. So altogether Theorem 4.4 can be interpreted as saying that the stable homotopy theory of spaces retractive over BG is equivalent to the homotopy theory of $\mathbb{S}[G]$-modules, or spectra with an action of G. This is exploited by Klein and Rognes to prove a chain rule for the Calculus of Functors [22].

7.3. Monoids and groups. The theories of sets, monoids and groups have equivalent stable homotopy theories. This follows from the fact (see [29, Theorem 1]) that the free monoid and the free group generated by a connected simplicial set are weakly equivalent to the loop space on the suspension of the simplicial set. Since the map from a simplicial set to the loop space of its suspension is twice as highly connected as the space itself, the maps of Gamma-rings

$$\mathbb{S} \to (\text{monoids})^s \to (\text{groups})^s$$

are stable equivalences.

7.4. Nilpotent groups. The lower central series of a group G is a filtration by normal subgroups $\Gamma_r G$. These subgroups are defined inductively by $\Gamma_1 G = G$ and $\Gamma_r G = [\Gamma_{r-1} G, G]$, the subgroup generated by commutators. A group is called nilpotent of class r if $\Gamma_{r+1} G$ is trivial. We denote by Nil^r the theory of class r nilpotent groups. We obtain a tower of theories

$$(\text{groups}) \to \cdots \to \text{Nil}^r \to \text{Nil}^{r-1} \to \cdots \to \text{Nil}^1 = (\text{abelian groups}).$$

It follows from a theorem of Curtis [12, Theorem 1.4] that the unit map $\mathbb{S} \to (\text{Nil}^r)^s$ is $(\log_2 r - 1)$-connected. So the associated sequence of Gamma-rings interpolates between $\mathbb{S}$ and $(\text{Nil}^1)^s = H\mathbb{Z}$.

7.5. p-local groups. Fix a prime number p. By considering p-local nilpotent groups we obtain a Gamma-ring model for the p-local sphere spectrum, together with a "multiplicative filtration".

This provides a different view at the mod p-lower central series spectral sequence of [8]. A nilpotent group G is called *p-local* if for all primes $q \neq p$ the set map $x \mapsto x^q$ is a bijection of G onto itself. On the category of nilpotent groups there exists a p-localization functor $G \mapsto G_{(p)}$ which is left adjoint to the inclusion of nilpotent p-local groups [41, Section 8]. p-localization is exact and commutes with the terms in the lower central series, i.e.,

$$(G/\Gamma_r G)_{(p)} \cong G_{(p)}/\Gamma_r(G_{(p)}).$$

p-local groups of fixed nilpotence class r form a theory which we denote by $\mathrm{Nil}^r_{(p)}$. By [10, Chapter IV] the group-theoretic localization map $G \to G_{(p)}$ induces p-localization on homotopy groups for every simplicial nilpotent group G. This implies that the map of Gamma-rings $(\mathrm{Nil}^r)^s \to (\mathrm{Nil}^r_{(p)})^s$ is the p-localization map on the associated spectra.

The category of simplicial theories has inverse limits and these are calculated pointwise [6, Proposition 3.11.1]. We denote by $\mathrm{Nil}^{\wedge}_{(p)}$ the inverse limit theory of the $\mathrm{Nil}^r_{(p)}$. If X is a reduced simplicial set, GX its Kan loop group, then the inverse limit of the simplicial groups $(GX/\Gamma_r(GX))_{(p)}$ is weakly equivalent to the loop group of the $\mathbb{Z}_{(p)}$-completion of X by [10, Chapter IV, Proposition 4.1]. In particular, the free $\mathrm{Nil}^{\wedge}_{(p)}$-algebra generated by a reduced simplicial set X is a model for the p-localization of $\Omega|\Sigma X|$. So the Gamma-ring $(\mathrm{Nil}^{\wedge}_{(p)})^s$ is a model for the p-local sphere spectrum.

We define J_i as the pointwise fibre of the map of Gamma-rings associated to $\mathrm{Nil}^{\wedge}_{(p)} \to \mathrm{Nil}^{p^i}_{(p)}$. Then

$$J_i \to (\mathrm{Nil}^{\wedge}_{(p)})^s \to (\mathrm{Nil}^{p^i}_{(p)})^s$$

is a homotopy fibre sequence of Γ-spaces: when evaluated at any simplicial set it gives a short exact sequence of simplicial groups. Since J_i is the fiber of a multiplicative map between Gamma-rings, it inherits a multiplication (but no unit), and it behaves like an ideal of $(\mathrm{Nil}^{\wedge}_{(p)})^s$. One can show that the J_i's even form a multiplicative filtration of $(\mathrm{Nil}^{\wedge}_{(p)})^s$, i.e., the image of $J_i \wedge J_j$ in $(\mathrm{Nil}^{\wedge}_{(p)})^s$ under the Gamma-ring multiplication is contained in J_{i+j}. Altogether we have obtained a convergent multiplicative filtration on a Gamma-ring model of the p-local sphere spectrum. This filtration in turn gives rise to a multiplicative spectral sequence. There is a variant which starts with the p-lower central series, and gives a multiplicative filtration on the p-completed sphere spectrum. In that case, the spectral sequence obtained from the filtration is the mod-p lower central series spectral sequence of [8]. From the E^2-term on this spectral sequence is the Adams spectral sequence.

7.6. Infinite loop spaces. In our simplicial setup, the Barratt–Eccles model [1,2] gives an algebraic theory modeling infinite loop spaces. Barratt and Eccles define a functor Γ^+ from the category of pointed simplicial sets to itself [1, Definition 3.1]. To avoid notational confusion with the category Γ^{op} of finite pointed sets, we use the notation γ^+ for the functor of Barratt and Eccles. The functor γ^+ is degreewise defined and commutes with filtered colimits, i.e., it comes from a Γ-space, and γ^+ has the structure of a triple [1, Proposition 3.6]. So γ^+ is the free algebra functor of a simplicial theory. This is in fact the only example of a simplicial theory which we consider explicitly and which is not a discrete theory. The algebras over this theory are called "simplicial set with Γ^+-structure" in [1]. For connected pointed simplicial sets X, $\gamma^+ X$ is a model for $\Omega^\infty\Sigma^\infty|X|$ (this is proved for Kan complexes in [1, Theorem 4.10, 5.4], but since the functor γ^+ is a prolonged Γ-space, it preserves weak equivalences of simplicial sets [9, Section 4.9] so that property holds for arbitrary X). Every γ^+-algebra X is naturally a simplicial monoid. Barratt and Eccles show furthermore [2, Theorem A] that if $\pi_0 X$ is a group, then the γ^+-structure provides natural infinite deloopings of X. In this sense, the algebraic theory γ^+ models infinite loop spaces. The Gamma-ring $(\gamma^+)^s$ arising from the theory γ^+ is yet another model for the sphere spectrum; it has the property that its underlying Γ-space is special.

7.7. Modules. Let B be a simplicial ring and consider the theory of simplicial left B-modules. The Gamma-ring obtained from this theory is the Eilenberg–MacLane Gamma-ring HB as defined in 1.5. The homotopy theory of B-modules remains unchanged under stabilization (cf. [36, Theorem 2.2.2]). Theorem 4.4 thus says that the homotopy theory of simplicial B-modules is equivalent to the homotopy theory of HB-modules; we recover [37, Theorem 4.4].

7.8. Associative algebras. Let B be a commutative simplicial ring and consider the theory of augmented associative B-algebras (alias associative B-algebras without unit). We claim that the map from the theory of B-modules to the theory of augmented associative B-algebras induces a weak equivalence on associated Gamma-rings

$$HB \xrightarrow{\sim} (\text{Ass. } B\text{-alg})^s.$$

The connective stable homotopy theory of augmented associative B-algebras is thus equivalent to the homotopy theory of simplicial B-modules. This fact could have been proven without ever introducing Gamma-rings by the methods of [36, Section 3]. To prove the claim we note that the free associative non-unital B-algebra generated by a pointed simplicial set K decomposes as the direct sum

$$\bigoplus_{n=1}^{\infty} \tilde{B}[\underbrace{K \wedge \cdots \wedge K}_{n}]$$

where $\tilde{B}[-]$ denotes the reduced free B-module. If K is taken to be a k-dimensional sphere, all homogeneous components of degree $\geqslant 2$ are at least $(2k-1)$-connected, so the map from the free B-module on S^k to the free non-unital associative B-algebra on S^k is $(2k-2)$-connected.

7.9. Commutative algebras. Let B be a commutative simplicial ring and consider the theory of augmented commutative B-algebras (alias commutative B-algebras without unit) Commutative simplicial algebras have been the object of much study [14,17,18,31,36]. The homology theory arising as the derived functor of abelianization in this case is known as André–Quillen homology for commutative rings.

We denote by DB the Gamma-ring arising from the theory of augmented commutative B-algebras. If B is a $\mathbb{Q}$-algebra, the map $HB \to DB$ induced from the symmetric algebra functor is a stable equivalence (cf. [36, Theorem 3.2.3]). In general, the Eilenberg–MacLane spectrum splits off DB, but the category of commutative augmented B-algebras can have higher stable homotopy operations, in which case DB is not equivalent to HB.

We claim that as a Γ-space, DB is stably equivalent to $HB \wedge^L H\mathbb{Z}$. In particular, the homotopy groups of DB are additively isomorphic to the integral spectrum homology of the Eilenberg–MacLane spectrum HB. To prove our claim we use that the free commutative B-algebra without unit on a pointed set X is additively isomorphic to the reduced free B-module on the infinite symmetric product of X. Hence if we let SP denote the Γ-space that sends a pointed set to the infinite symmetric product, then DB is isomorphic to the composite Γ-space $HB \circ \mathrm{SP}$. For every connected simplicial abelian monoid the map to its group completion is a weak equivalence [40, Corollary 5.7], so the group completion map $\mathrm{SP} \to H\mathbb{Z}$ is a stable weak equivalence of Γ-spaces. Hence DB is weakly equivalent to the derived smash product of HB and $H\mathbb{Z}$ by [25, Proposition 5.23]. $HB \wedge^L H\mathbb{Z}$ can be constructed as an E_∞-ring spectrum, but the weak equivalence to DB cannot be multiplicative in any sense since the ring of homotopy groups of $HB \wedge^L H\mathbb{Z}$ is graded commutative, that of DB is generally not.

By Section 4.11 the ring $\pi_* DB$ is isomorphic to the ring of stable homotopy operations of commutative simplicial B-algebras. These operations are also referred to as the stable Cartan–Bousfield–Dwyer algebra (since these authors calculated the unstable operations for $B = \mathbb{F}_p$, see [11,7,14]). Additively, $\pi_* DB$ is the direct sum of the stable derived functors, in the sense of Dold and Puppe [13, Section 8.3], of the symmetric power functors on the category of B-modules. In [7, Section 12], Bousfield calculates the ring $\pi_* D\mathbb{F}_p$ under the name of "stable algebra of the functor algebra" of symmetric powers. For $p = 2$ ([7, Theorem 12.3]; see also [14]) it is the associative unital $\mathbb{F}_2$-algebra with generators α_i for $i \geqslant 2$ subject to the relations

$$\sum_{i+j=n} \binom{n}{i} \alpha_{2m+i}\alpha_{1+m+j} = 0 \quad \text{and} \quad \sum_{i+j=n} \binom{n}{i} \alpha_{1+2m+i}\alpha_{1+m+j} = 0$$

for $m, n \geqslant 0$. [7, Theorem 12.6] gives a similar but more complicated description for odd primes.

If X is an augmented commutative simplicial B-algebra, its stable homotopy is defined as the homotopy groups of the suspension spectrum of any cofibrant replacement: $\pi_*^s X = \pi_* \Sigma^\infty X^c$. Then the stable homotopy and André–Quillen homology of X are related by the universal coefficient and Atiyah–Hirzebruch spectral sequences of Section 5.5

$$\mathrm{Tor}_p^{\pi_* DB}(\pi_* B, \pi_*^s X)_q \Rightarrow \mathrm{H}_{p+q}^{AQ} X$$

$$\mathrm{H}_p^{AQ}(X; \pi_q DB) \Rightarrow \pi_{p+q}^s X.$$

7.10. Divided power and Lie algebras. In the spirit of the previous two examples, one can consider other types of algebras over a commutative ring B and study the Gamma-rings they give rise to. For divided power algebras over $\mathbb{F}_p$, Bousfield [7, Theorems 12.3 and 12.6] calculates the graded ring of homotopy groups of the associated Gamma-ring, again under the name of "stable derived functors" of the divided power functors. For the case of restricted Lie algebras over $\mathbb{F}_p$, this calculation is carried out in [8, Theorems 2.4 and 2.4']. The result is known as the Λ-algebra, and it shows up as a E^1-term of the Adams spectral sequence for the stable homotopy groups of spheres. The case of restricted Lie algebras is closely related to Example 7.5 since the associated graded to the p-lower central series filtration of a free group is the free restricted Lie algebra on the abelianized group.

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Appendix A. Cofibrantly generated model categories

In [32, II.p.3.4], Quillen formulates his small object argument, which is now a standard device for producing model category structures. An example of this is the lifting Lemma A.2 below which we use several times in this paper. After Quillen, various other authors have axiomatized and generalized the small object argument. We work with the "cofibrantly generated model categories" of [15]. We have given a review of cofibrantly generated model categories in [37, Appendix A] and we will continue to use that terminology. In all the cases we treat in this paper, category theory automatically takes care of the smallness conditions. The basic reason is that we are dealing with suitable functor categories with values in simplicial sets. The relevant category theoretical notion is that of a locally presentable category. The categories we consider are even locally finitely presentable. In general, categories involving actual topological spaces tend *not* to be locally presentable.

An object K of a category $\mathscr{C}$ is called *finitely presentable* if the hom functor $\hom_{\mathscr{C}}(K,-)$ preserves filtered colimits. A set $\mathscr{G}$ of objects of a category $\mathscr{C}$ is called a set of *strong generators* if for every object K and every proper subobject there exists $G \in \mathscr{G}$ and a morphism $G \to K$ which does not factor through the subobject. A category is called *locally finitely presentable* if it is cocomplete and has a set of finitely presentable strong generators.

A.1. Lemma. *Let T be a simplicial theory. Then the categories T-alg, $\mathscr{GS}(T)$ and $\mathscr{S}p(T)$ are locally finitely presentable. If T is a discrete theory and R a ring, then the category $s\mathscr{F}(T^{\mathrm{op}},R)$ of simplicial functors is locally finitely presentable. If S is a Gamma-ring, then the category of S-modules is locally finitely presentable.*

Proof. All the above categories are cocomplete. Finitely presentable strong generators exist because objectwise evaluation is representable in all these categories. More precisely, possible choices of generators are as follows. In T-alg, we can choose the T-algebras freely generated by the simplicial standard simplices $(\Delta^i)^+$. In $\mathscr{GS}(T)$ we can take the objects $T^s \circ (\Gamma^n \wedge (\Delta^i)^+)$. In $\mathscr{S}p(T)$ we take the spectra of T-algebras F_n^i defined by

$$(F_n^i)_j = \begin{cases} * & \text{if } j < n, \\ \Sigma^{j-n} F^T((\Delta^i)^+) & \text{if } j \geqslant n. \end{cases}$$

In the category $s\mathscr{F}(T^{\mathrm{op}}, R)$ of simplicial functors the objects $P_n \otimes \mathbb{Z}[\Delta^i]$ serve as generators, where P_n denotes the projective generator associated to the object n^+ of T^{op} as in Section 6.1. Finally, for a Gamma-ring S the modules $S \wedge \Gamma^n \wedge (\Delta^i)^+$ do the job. $\square$

We make no claim to originality for the following lifting lemma; various other lifting lemmas can be found in the model category literature. We use the lifting lemma to obtain model category structures for the categories listed in Lemma A.1. A proof of the lemma is given in [37, B.2 and B.3]. Let $\mathscr{C}$ be a simplicial model category, and let $\mathscr{D}$ be a complete and cocomplete category which is tensored and cotensored over the category of simplicial sets. Consider a simplicial adjoint functor pair

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}.$$

By this we mean that L and R have the structure of simplicial functors and the adjunction extends to an isomorphism of simplicial hom sets. We call a map $f: X \to Y$ in $\mathscr{D}$ a weak equivalence (resp. fibration) if the map $R(f): R(X) \to R(Y)$ is a weak equivalence (resp. fibration) in $\mathscr{C}$. A map in $\mathscr{D}$ is called a cofibration if it has the left lifting property with respect to all acyclic fibrations.

A.2. Lemma. ([37, B.2 and B.3]). *In the above situation, assume that the model category $\mathscr{C}$ is cofibrantly generated and that the category $\mathscr{D}$ is locally finitely presentable. Assume further that there exists a functor $Q: \mathscr{D} \to \mathscr{D}$ and a natural weak equivalence $X \to QX$ such that QX is fibrant for all X. Then the category $\mathscr{D}$ becomes a cofibrantly generated closed simplicial model category.*

The last lemma about the Bousfield–Friedlander category of spectra of [9, Theorem 2.3] is needed in Theorem 4.3 to obtain the model category structure for spectra of algebras over an algebraic theory.

A.3. Lemma. *The stable model category structure for spectra is cofibrantly generated.*

Proof. The functor that takes a spectrum to its nth term has a left adjoint which we will denote F_n. F_nK is a shift desuspension of the suspension spectrum defined by

$$(F_nK)_j = \begin{cases} * & \text{if } j < n, \\ S^{j-n} \wedge K & \text{if } j \geqslant n. \end{cases}$$

There are three types of generating (acyclic) cofibrations:

(A) $F_n(\partial\Delta^i)^+ \to F_n(\Delta^i)^+$

(B) $F_n(\Lambda^{i,k})^+ \xrightarrow{\sim} F_n(\Delta^i)^+$

(C) $(F_{n+j}S^j) \wedge (\Delta^i)^+ \cup_{(F_{n+j}S^j) \wedge (\partial\Delta^i)^+} C_n^j \wedge (\partial\Delta^i)^+ \xrightarrow{\sim} C_n^j \wedge (\Delta^i)^+.$

Here Δ^i denotes the simplicial i-simplex, $\partial\Delta^i$ its boundary and $\Lambda^{i,k}$ the kth horn. Also

$$C_n^j = (F_{n+j}S^j \wedge (\Delta^1)^+) \cup_{F_{n+j}S^j \times 1} F_nS^0$$

is the reduced mapping cylinder of the map $F_{n+j}S^j \to F_nS^0$ which is the identity in spectrum degrees above $n+j$ (this map is thus a stable weak equivalence, but it is not a cofibration). Since both source and target of this map are cofibrant, the inclusion of the source into the mapping cylinder is a cofibration. The projection $C_n^j \to F_n S^0$ is a simplicial homotopy equivalence.

The first observation is that a map of spectra $X \to Y$ has the right lifting property for maps of type (A) if and only if all maps $X_n \to Y_n$ are acyclic fibrations of simplicial sets. So right lifting property for maps of type (A) is equivalent to being a strict acyclic fibration. Since cofibrations coincide in the strict and stable model category structures, so do the acyclic fibrations. Hence the maps of type (A) qualify as generating cofibrations. Similarly, right lifting property for all maps of type (B) is equivalent to being a strict fibration. We claim that if in addition the right lifting property for maps of type (C) holds, then the map $X \to Y$ is a stable fibration. This shows that the maps of type (B) and (C) qualify as generating acyclic cofibrations.

To prove the claim we first show that if a strict fibration of spectra $X \to Y$ has the right lifting property with respect to the maps of type (C), then the maps $X_n \to Y_n \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j}$ are weak equivalences for all n and j. Note that $\mathrm{hom}(F_k S^l, X) \cong \Omega^l X_k$ where the left-hand side denotes simplicial hom sets of spectra and where Ω denotes the simplicial set of maps from the simplicial circle S^1 (which can have the "wrong" homotopy type if the argument is not fibrant). The right lifting property with respect to the maps of type (C) is thus equivalent to the maps

$$\mathrm{hom}(C_n^j, X) \to \mathrm{hom}(C_n^j, Y) \times_{\mathrm{hom}(F_{n+j}S^j, Y)} \mathrm{hom}(F_{n+j}S^j, X) = \mathrm{hom}(C_n^j, Y) \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j}$$

being acyclic fibrations of simplicial sets. Since $C_n^j \to F_n S^0$ is a simplicial homotopy equivalence, so is the map $X_n \cong \mathrm{hom}(F_n S^0, X) \to \mathrm{hom}(C_n^j, X)$. In the commutative diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "\sim"'] & Y_n \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j} \arrow[r] \arrow[d, "\sim"] & Y_n \arrow[d, "\sim"] \\
\mathrm{hom}(C_n^j, X) \arrow[r] & \mathrm{hom}(C_n^j, Y) \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j} \arrow[r] & \mathrm{hom}(C_n^j, Y)
\end{tikzcd}$$

the right square is a pullback square in which the lower horizontal map is a fibration and the right vertical map is a weak equivalence. So the middle vertical map is a weak equivalence, and so is the map $X_n \to Y_n \times_{\Omega^j Y_{n+j}} \Omega^j X_{n+j}$.

Since $X_{n+j} \to Y_{n+j}$ is a fibration of simplicial sets, the right square in the commutative diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & \Omega^j X_{n+j} \arrow[r] \arrow[d] & \Omega^j \mathrm{Sing}\,|X_{n+j}| \arrow[d] \\
Y_n \arrow[r] & \Omega^j Y_{n+j} \arrow[r] & \Omega^j \mathrm{Sing}\,|Y_{n+j}|
\end{tikzcd}$$

is homotopy cartesian (even though the horizontal maps need not be weak equivalences). By what we proved in the previous paragraph the left square is also homotopy cartesian, so the outer composed square is homotopy cartesian for all n and j. It follows that the squares

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (QX)_n \arrow[d] \\ Y_n \arrow[r] & (QY)_n \end{tikzcd}$$

are homotopy cartesian where $(QX)_n = \operatorname{colim}_j \Omega^j \operatorname{Sing}|X_{n+j}|$. So the fibration criterion [9, Section A.7] is satisfied and the map $X \to Y$ is a stable fibration of spectra. □

References

[1] M.G. Barratt, P.J. Eccles, Γ^+-structures I: a free group functor for stable homotopy theory, Topology 13 (1974) 25–45.

[2] M.G. Barratt, P.J. Eccles, Γ^+-structures II: a recognition principle for infinite loop spaces, Topology 13 (1974) 113–126.

[3] J. Beck, On H-spaces and infinite loop spaces, in Category Theory, Homology Theory and Their Applications III, Lecture Notes in Mathematics, Vol. 99, Springer, Berlin, 1969, pp. 139–153.

[4] J.M. Boardman, R.M. Vogt, Homotopy Invariant Algebraic Structures on Topological Spaces, Lecture Notes in Mathematics, Vol. 347, Springer, Berlin, 1973.

[5] M. Bökstedt, Topological Hochschild homology, Preprint, Bielefeld, 1985.

[6] F. Borceux, Handbook of Categorical Algebra 2: Categories and Structures, Encyclopedia of Mathematics and its Applications, Vol. 51, Cambridge University Press, Cambridge, 1994.

[7] A.K. Bousfield, Operations on derived functors of non-additive functors, Mimeographed Notes, Brandeis University, 1967.

[8] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, J.W. Schlesinger, The mod-p lower central series and the Adams spectral sequence, Topology 5 (1966) 331–342.

[9] A.K. Bousfield, E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in Geometric Applications of Homotopy Theory II, Lecture Notes in Mathematics, Vol. 658, Springer, Berlin, 1978, pp. 80–130.

[10] A.K. Bousfield, D.M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer, Berlin, 1972.

[11] H. Cartan, Algébres d'Eilenberg–MacLane et homotopie, Seminaire Henri Cartan, 1954–55.

[12] E.B. Curtis, Some relations between homotopy and homology, Annals of Mathematics 83 (2) (1965) 386–413.

[13] A. Dold, D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen, Annales de l'Institut Fourier Grenoble 11 (1961) 201–312.

[14] W.G. Dwyer, Homotopy operations for simplicial commutative algebras, Transactions of the American Mathematical Society 260 (1980) 421–435.

[15] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, Model categories and more general abstract homotopy theory, in preparation.

[16] W.G. Dwyer, J. Spalinski, Homotopy theories and model categories, in: I.M. James (Ed.), Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[17] P.G. Goerss, A Hilton-Milnor theorem for categories of simplicial algebras, American Journal of Mathematics 111 (1989) 927–971.

[18] P.G. Goerss, On the André-Quillen cohomology of commutative $\mathbb{F}_2$-algebras, Asterisque 186 (1990) 169.

[19] J.F. Jardine, The homotopical foundations of algebraic K-theory. In Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemporary Mathematics, Vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 57–82.

[20] M. Jibladze, T. Pirashvili, Cohomology of algebraic theories, Journal of Algebra 137 (1991) 253–296.

[21] D.M. Kan, On c.s.s. complexes, American Journal of Mathematics 79 (1957) 449–476.
[22] J.R. Klein, J. Rognes, A chain rule in the calculus of homotopy functors, Preprint, 1996.
[23] F.W. Lawvere, Functorial semantics of algebraic theories, Proceedings of the National Academy of Sciences U.S.A. 50 (1963) 869–872.
[24] F.E.J. Linton, Coequalizers in categories of algebras, in Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Lecture Notes in Mathematics, Vol. 80, Springer, Berlin, 1969, pp. 75–90.
[25] M. Lydakis, Smash products and Γ-spaces, Mathematical Proceedings of the Cambridge Philosophical Society 126 (1999) 311–328.
[26] S. MacLane, Homologie des anneaux et des modules, in Colloque de topologie algébrique, Louvain, 1956. Georges Thone, Liège, 1957, pp. 55–80.
[27] S. MacLane, Categories for the Working Mathematician, Springer, Berlin, 1971.
[28] J.P. May, The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 271, Springer, Berlin, 1972.
[29] J.W. Milnor, On the construction FK, in: J.F. Adams (Ed.), Algebraic Topology—a Student's Guide, Cambridge University Press, London, 1972, pp. vi + 300. London Mathematical Society Lecture Note Series, No. 4.
[30] T. Pirashvili, F. Waldhausen, MacLane homology and topological Hochschild homology, Journal of Pure and Applied Algebra 82 (1992) 81–98.
[31] D. Quillen, On the (co-) homology of commutative rings, in Applications of Categorical Algebra (Proceedings of Symposium on Pure Mathematics, Vol. XVII, New York, 1968). Amer. Math. Soc. Providence, R.I., 1970, pp. 65–87.
[32] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, Vol. 43, Springer, Berlin, 1967.
[33] C.L. Reedy, Homology of algebraic theories, Ph.D. Thesis, University of California, San Diego, 1974.
[34] R. Schwänzl, R.M. Vogt, The categories of A_∞- and E_∞-monoids and ring spaces as closed simplical and topological model categories, Archives of Mathematics 56 (1991) 405–411.
[35] R. Schwänzl, R.M. Vogt, Homotopy ring spaces and their matrix rings, in Proceedings of Algebraic Topology Poznan, 1989, Lecture Notes in Mathematics, Vol. 1474, Springer, Berlin, 1991, pp. 254–272.
[36] S. Schwede, Spectra in model categories and applications to the algebraic cotangent complex, Journal of Pure and Applied Algebra 120 (1997) 77–104.
[37] S. Schwede, Stable homotopical algebra and Γ-spaces, Mathematical Proceedings of the Cambridge Philosophical Society 126 (1999) 329–356.
[38] G. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.
[39] B. Shipley, Symmetric spectra and topological Hochschild homology, to appear, K-Theory.
[40] E. Spanier, Infinite symmetric products, function spaces, and duality, Annals of Mathematics (2) 69 (1959) 142–198[erratum, 733].
[41] R.B. Warfield, Nilpotent Groups, Lecture Notes in Mathematics, Vol. 513, Springer, Berlin, 1976.