Universal Symmetries: Global Equivariant Homotopy Theory

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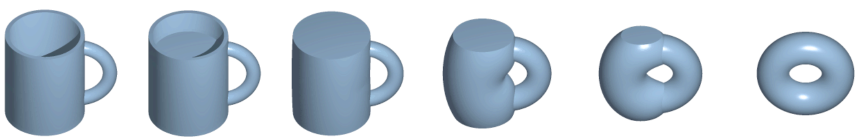
Abstract. Global equivariant homotopy theory is, informally speaking, the universal home for deformations of geometric structures with symmetry groups that vary coherently. It studies geometric and topological objects on which all compact Lie groups act at once and in a compatible and coherent way. In this survey article, we give a brief introduction to the subject with emphasis on recent applications to equivariant bordism.

1. Introduction

One important aim of algebraic topology is the classification of nice spaces, such as manifolds or cell complexes, up to deformation, or more precisely up to homotopy equivalence:



A major tool to distinguish different homotopy types are algebraic invariants, such as the fundamental group, higher homotopy groups, or cohomology theories. Some of the historically first and particularly prominent (co)homology theories are

- singular cohomology $H^n(X;\mathbb{Z})$, made from 'singular simplices' in the space X, i.e., continuous maps $\Delta^n \longrightarrow X$ from the standard simplices;
- the bordism groups $\mathcal{N}_n(X)$, where elements are represented by continuous maps $M^n \longrightarrow X$ from smooth closed manifolds of dimension n, taken up to bordism;
- topological K-theory $K(X)$, where elements are formal differences of vector bundles over the space X.

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[1]mug and torus morph: public domain, original gif-file created by Lucas Vieira. https://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif

A major milestone in algebraic topology was the insight in the late 1950s that homology theories and cohomology theories are represented by *spectra*. For example, the (co)homology theories just mentioned are represented:

- Singular (co)homology is represented by the *Eilenberg–MacLane spectrum*, often denoted $H\mathbb{Z}$.
- The bordism homology theory is represented by the *Thom spectrum*, usually denoted MO. Popular variations of this theme endow the manifolds and bordisms with additional tangential or normal structure; these bordism homology theories are represented by spectra such as MSO (for oriented manifolds) or MU (for stably almost complex manifolds).
- Bott periodicity allows one to extend the Grothendieck rings of vector bundles to a multiplicative cohomology theory, represented by the *K-theory spectrum KU*.

The main advantage of spectra over (co)homology theories is that they are more flexible and easier to manipulate.

But: What are spectra? Well, the answer varies, depending on when the question is asked, and who is being asked. A crude approximation to this concept is motivated by the feature that cohomology theories come with suspension isomorphisms, i.e., suspending a space shifts the reduced (co)homology groups. So one should obtain spectra from spaces by formally inverting the suspension operation. The crux lies in how 'inversion of suspension' is implemented. First passing to the homotopy category of based spaces and then formally inverting the suspension functor yields the *Spanier–Whitehead category* [44]. This category has certainly been useful, and when restricted to finite CW-complexes it correctly models the homotopy category of compact spectra (sometimes also called 'small' or 'finite' spectra, the thick subcategory generated by the sphere spectrum).

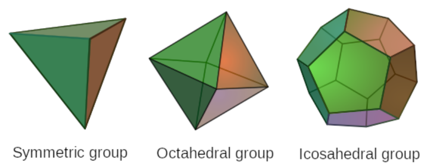
In the early days of the subject, the only *raison d'être* of spectra was to represent (co)homology theories, and any definition that fit this purpose was fine. To represent homology theories and cohomology theories is still an important purpose of spectra to this day. Over time, the entirety of spectra was more and more studied for its own sake, and the collection of all spectra was organized into increasingly more refined mathematical structures. Until the 1990's, most stable homotopy theorists would probably have argued that spectra form a *triangulated category*. A spectrum would be some kind of sequence of based spaces connected by continuous maps from the suspension of one space to the next space; but getting the morphisms right is a bit tricky. The first fully functional category of spectra was constructed by Boardman in his thesis [5], and Adams' exposition [1, Part III] was particularly popular for a long time.

Then until the mid 2000's, *model categories* were the predominant concept for organizing spectra, and some (including the author) made a living by constructing and comparing model categories of spectra. I my opinion, the advent of higher category theory greatly clarified what spectra really *are* from a conceptual point of view: they are the free presentable stable ∞-category on one object (the sphere spectrum), see [33, Corollary 1.4.4.6]. In the realm of triangulated categories, there is no such universal property for the stable homotopy category. And while the universal property can in principle be expressed in the language of model categories, it is much more clumsy to implement. For example, [36, Theorem 5.1] amounts to saying that a certain moduli space of left Quillen functors is nonempty and connected, but it falls short of showing that this space is actually contractible. Since the ∞-category of spaces (or, as some prefer, of 'homotopy types', '∞-groupoids', or 'animae') is the free presentable ∞-category on one object (the contractible anima), spectra indeed are the ∞-categorical stabilization of spaces. And since 'stable' essentially means that suspension is invertible, in this precise sense spectra are obtained from spaces by inverting suspension.

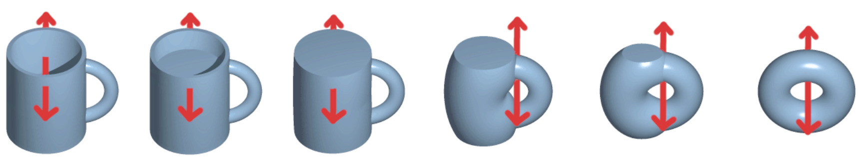
2. Equivariant spectra

Interesting mathematical objects that occur 'in nature' tend to have symmetries; and who would not agree that embracing the natural symmetries is beneficial? For my story, the symmetry groups are either finite groups or, more generally, compact Lie groups.

[2]



An important aim of equivariant algebraic topology is to classify certain kinds of equivariant spaces or manifolds up to symmetry preserving deformation (equivariant homotopy equivalence):



Again, algebraic invariants such as equivariant cohomology theories come in handy to tell apart different equivariant homotopy types. In turns out that many classical (co)homology theories have equivariant refinements:

- Bredon (co)cohomology [9] is a G-equivariant extension of singular cohomology; its coefficients are not merely abelian groups, but rather 'coefficient systems', i.e.,

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`https://commons.wikimedia.org/wiki/File:Tetrahedron.svg`
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co- or contravariant functors from the G-orbit category to abelian groups. The group G can be very general here, such as an arbitrary topological group [25].

- The equivariant bordism group $\mathcal{N}_n^G(X)$ consists of G-bordism classes of continuous G-maps $M^n \longrightarrow X$, where the source is now a smooth closed G-manifold of dimension n. To actually get a homology theory, restrictions on the group are necessary. Compact Lie groups are fine; and general Lie groups (not necessarily compact) work if one restricts to proper actions. I will return to equivariant bordism in more detail in Section 4.
- Equivariant K-theory $K_G(X)$ works nicely for compact Lie groups of equivariance; for compact G-spaces, elements are again formal differences of G-equivariant vector bundles over X.
- Every non-equivariant cohomology theory F^* can be turned into an equivariant cohomology theory F^*_G by applying F^* to the homotopy orbit construction. In other words, for a G-space X, one simply sets $F^*_G(X) = F^*(EG \times_G X)$, where EG is a universal free G-space, of the G-homotopy type of a G-CW-complex. The homotopy orbit construction is also known as the 'Borel construction', and hence F^*_G also goes under the name of *Borel cohomology*.

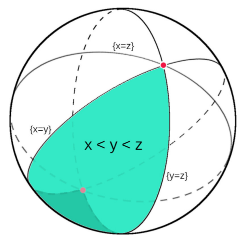
Now you can probably guess what is coming next: G-equivariant cohomology theories are represented by objects known as *G-spectra*, which can be obtained from based G-spaces by 'inverting suspension'. However, in the equivariant situation new subtleties arise, and the term 'inverting suspension' has even more possible interpretations. Depending on the flavor of suspensions to be inverted, the resulting objects are known as either *naive* or *genuine* G-spectra.

For our purposes, the 'good' class of G-spaces are the G-CW-complexes, which we study up to G-homotopy equivalence. For compact Lie groups, Illman's triangulation theorem [26] ensures that smooth G-manifolds belong to this class. One can get the same homotopy theory by working with general G-spaces, but then the correct notion of equivalence is that of an equivariant continuous map that induces a weak homotopy equivalence on all fixed points for all closed subgroups. Elmendorf's theorem [18] identifies this 'fine' homotopy theory of G-spaces with continuous functors from the G-orbit category to spaces. It became somewhat popular recently to even *define* G-spaces as functors from the orbit category to the ∞-category of spaces. While this is mathematically legitimate, degrading G-spaces to just another presheaf category makes me melancholic... I would like a G-space to be some kind of geometric or topological object with symmetries parameterized by G. But I might just be old-fashioned.

But now back to G-spectra. For any topological group, there is always a good notion of G-spectra; and for some classes of groups such as compact Lie groups, there is an even better notion of G-spectra. The good version of G-spectra – often called 'naive' – is anything that models the stabilization, in the sense of higher categories, of the fine homotopy theory of G-spaces. So naive G-spectra are obtained from based G-spaces by inverting suspension; they represent (co)homology theories with a suspension isomorphism for ordinary suspensions. (There is also a 'doubly naive' version of G-spectra, namely functors, in the ∞-categorical sense, from the classifying space BG to spectra. But let's not go into that.)

The even better version of G-spectra, called *genuine*, is only available for special classes of symmetry groups. For me, the class of compact Lie groups is the preferred setting; special things happen for the subclass of finite groups, which might be why some like to restrict to that context. I do believe that for phenomena that involve representations, equivariant vector bundles, or equivariant bordism, it is a bad idea to restrict attention to finite groups, because the universal instances of important classes, such as Thom classes, Euler classes or Chern classes, tend to live at orthogonal or unitary groups.

So what is the difference between naive and genuine G-spectra? Succinctly: to get genuine spectra, we also invert all 'twisted suspensions', i.e., smashing with linear representation spheres. For this purpose, a *G-representation* is a finite-dimensional $\mathbb{R}$-vector space V on which G acts by linear isometries; and the *representation sphere* S^V is the onepoint compactification, based at infinity. On the right is a picture of the unit sphere of the natural Σ_3-representation on $\mathbb{R}^3$ by permuting the coordinates. It is simultaneously a picture of the onepoint compactification of the reduced natural Σ_3-representation, permuting the coordinates of $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$.



Inverting all linear representations spheres has some benefits over inverting only spheres with trivial G-action. For example, cohomology theories represented by genuine G-spectra can be extended from integer-graded objects to $RO(G)$-graded objects [30]. I must issue the warning, however, that properly setting up an $RO(G)$-grading is a slippery slope, in particular with respect to multiplicative structures. Readers unfamiliar with the subtleties and pitfalls of $RO(G)$-gradings are encouraged to peruse Adams' sardonic and timelessly entertaining introduction to [2, §6]. Incidentally, this is the only paper I know of that is referenced in MathSciNet, and which references the bible, specifically the book Genesis. Two other advantages of genuine G-spectra over naive G-spectra are better duality and transfer properties. In genuine G-spectra, finite G-CW-complexes become dualizable, and Atiyah duality holds for smooth compact G-manifolds. Transfers, also known as 'pushforward', 'wrong way' or 'Umkehr' maps, exists for equivariant fibrations whose fibers are finite CW-complexes. Special instances of the transfers provide the equivariant homotopy groups with the structure of a Mackey functor. For finite groups G, every genuine G-spectrum has a natural filtration whose subquotients 'are' G-Mackey functors. More formally, the heart of the most obvious t-structure on the genuine G-stable category is equivalent to the abelian category of G-Mackey functors.

Genuine G-spectra have been generalized beyond the class of compact Lie groups, most notably to profinite groups and to non-compact Lie groups [16, 19]; in these cases, the 'genuineness' is inherited from the finite quotients or the compact subgroups, respectively. One could contemplate inverting even more kinds of 'equivariant spheres', arising for example from homotopy-representations. But this has no additional effect: every based finite G-CW complex all of whose fixed point spaces, for all closed subgroups of G, are homology spheres, is already invertible in the genuine G-stable homotopy category. But there are variations that interpolate between 'doubly naive', 'naive' and 'genuine' by varying some of the available parameters: we could decide to test equivalences on fixed points for some, but not all, subgroups; and we could invert some, but not all, linear representation spheres. In my opinion, out of the zoo of different brands of G-spectra available, the genuine G-spectra are the richest and most interesting.

The cohomology theories mentioned above are representable by genuine G-spectra:

- For finite groups G, Bredon (co)homology with coefficients in a G-Mackey functor is represented by an associated *Eilenberg–MacLane G-spectrum*. Mind the fact that while Bredon homology and cohomology only need, respectively, a covariant or contravariant coefficient system, we need a G-Mackey functor to be representable by a genuine G-spectrum. This should not be surprising given that the equivariant homotopy groups of a genuine G-spectrum always come with this enhanced algebraic structure.
- The homology theory of G-equivariant bordism is represented by a *G-Thom spectrum* $\mathbf{mO}^G$. Well, not quite, but at least for a large class of groups. More precisely, for every compact Lie group G, the equivariant Thom–Pontryagin construction is a natural transformation from geometric bordism $\mathcal{N}_*^G(X)$ to the equivariant homology theory $\mathbf{mO}_*^G(X)$. And whenever G is isomorphic to a product of a finite group and a torus, then this map is an isomorphism. This result is usually credited to Wasserman because it can be derived from his equivariant transversality theorem [47, Theorem 3.11]; a homotopy theoretic proof is given in [38, Theorem 6.2.33]. The Thom–Pontryagin map is provably not bijective for more general compact Lie groups. For example, for $G = SU(2)$, a homotopy theoretic transfer in $\mathbf{mO}_0^{SU(2)}$ from the maximal torus normalizer to $SU(2)$ is not in the image, compare [38, Remark 6.2.34]. If we add additional tangential or normal structure to the picture, such as orientations or stably almost complex structures, it is not clear to me for which groups an appropriately adjusted Thom–Pontryagin map is bijective.

- *Stable equivariant bordism* $\mathfrak{N}_*^{G:S}(X)$ is a certain localization of equivariant bordism, obtained by formally inverting bordism classes associated to orthogonal G-representations. By a theorem of Bröcker and Hook [10], the homology theory $\mathfrak{N}_*^{G:S}(X)$ is represented, without any restriction on the compact Lie group, by another equivariant refinement $\mathbf{MO}^G$ of the classical bordism spectrum, a genuine G-spectrum known as *homotopical equivariant bordism*.
- Equivariant K-theory is represented by a genuine G-spectrum KU_G. This theory enjoys Bott periodicity for equivariant Spin^c-representations, and more generally Thom isomorphisms for equivariant Spin^c-vector bundles, the so-called *Atiyah–Bott–Shapiro* orientation.
- Every non-equivariant theory cohomology theory F^* is represented by a spectrum F. The associated Borel cohomology theory F^*_G is then represented by a genuine G-spectrum whose underlying spectrum is F, and on which G acts, in a certain precise sense, 'cofreely'.

There are always many different equivariant forms of any given non-equivariant spectrum. And as the two equivariant versions $\mathbf{mO}^G$ and $\mathbf{MO}^G$ of the classical Thom spectrum indicate, often several of the many available refinements are useful. Finding the 'best' equivariant form of a non-equivariant theory can be more of an art than a science.

Now that we discussed what genuine G-spectra do for us, what *are* they? Again, there are several legitimate answers that have evolved over time in much the same way as in the non-equivariant context, with 1-categorical models [31, 34] as the setup of choice for several decades, and with higher categorical constructions becoming increasingly more popular in recent years. The idea of genuine G-spectra as the 'equivariant stabilization' of the fine homotopy theory of G-spaces has been implemented in different ways. One model independent description works for all compact Lie groups: genuine G-spectra are the initial example of a presentably symmetric monoidal ∞-category equipped with a symmetric monoidal left adjoint from based G-spaces that inverts all linear representation spheres, see [21, Appendix C]; the universal functor is the equivariant suspension spectrum functor.

For finite groups, several other descriptions or characterizations of genuine G-spectra are available. If we work parameterized over the ∞-topos of genuine G-spaces, than there is an enhanced notion of 'stability' that includes the usual stability (i.e., suspension is invertible), and also a condition making precise that 'indexed sums are equivalent to indexed products'. Genuine G-spectra are then the free G-presentable G-stable G-∞-category on one object, compare [13, Theorem 9.13]. Moreover, the preferred t-structure on the genuine G-stable category with G-Mackey functors as its heart is a shadow of a much finer relationship: genuine G-spectra are equivalent to 'spectral Mackey functors' [4], i.e., additive functors to spectra from the ∞-category of spans of finite G-sets; see [11, Appendix A] for a streamlined proof.

3. Global spectra

All the examples of G-equivariant cohomology theories and G-equivariant spectra that I mentioned in the previous section occur 'uniformly for all groups'. This is always a clear sign that these theories are underlying a *globally-equivariant* cohomology theory, or a *global* spectrum. Of course there are also many useful G-equivariant spectra that are specific to one group G, but since those do not support my narrative, I did not mention any examples.

What are global spectra? Well, this question again has more than one useful answer.

The orthogonal spectrum model. If the question 'What are global spectra?' refers to a mathematically rigorous definition, my answer is: global spectra are the localization of the category of orthogonal spectra at the class of global equivalences defined in [38, Definition 4.1.3]. Here the reader may interpret 'localization' in terms of ∞-categories, or in the more classical sense of 1-categories. This definition of global spectra was proposed in January 2013, when I uploaded the first preprint version of the book [38] to my homepage. With the package came a model structure complementing the global equivalences, and the proof that the global stable homotopy category is tensor-triangulated and compactly generated by the suspension spectra of the global classifying spaces of all compact Lie groups. A somewhat differently looking definition of global spectra had been independently proposed by Anna Marie Bohmann in her thesis from 2011 [6, Chapter 4]. This part of Bohmann's thesis was later published as [7]; in Theorem 6.2 of that paper, Bohmann also shows that her 1-category of $\mathcal{I}$-spaces is equivalent to the category of orthogonal spectra. Bohmann does not discuss global equivalences.

The orthogonal spectrum model for global spectra is not only historically the first approach, there are still some features that to my knowledge, and at the time of this writing, can only be implemented in this setup. One example is the full theory of multiplicative ultra-commutativity, both stably and unstably, when taking all compact Lie groups into account. Another example is that without Joachim's orthogonal spectrum model for global equivariant K-theory [27], I would not know how to establish its rich structure as an ultra-commutative global ring spectrum.

But possibly the question 'What are global spectra?' really meant: How should one think of global spectra in conceptual and model-independent terms? Then, too, there are several possible answers. Before we explain what global spectra are conceptually, I hasten to clarify what they are not: global spectra are *not* the ∞-categorical stabilization of global spaces. In contemporary higher categorical notation,

$$\mathcal{S}p_{\mathrm{gl}} \not\simeq \mathcal{S}p \otimes \mathcal{s}pc_{\mathrm{gl}} ,$$

i.e., global spectra are not Lurie's tensor product, in presentable ∞-categories, of the ∞-category $\mathcal{S}p$ of spectra and the ∞-topos $\mathcal{s}pc_{\mathrm{gl}}$ of global spaces.

Global spectra are compatible families of equivariant spectra. I suspect that the idea of global spectra as 'coherently compatible families of equivariant spectra' predates all formal definitions by quite a bit, and might have already been around in the early days of equivariant stable homotopy theory. Practitioners of the subject were certainly well aware that several important families of equivariant cohomology theories, such as equivariant K-theory, Bredon homology, Borel homology, equivariant stable cohomology, and equivariant bordism, naturally come in 'coherent families'.

The earliest published reference of a formal definition that I know of is by Lewis and May in [31, Chapter II, Definition 8.5]. Their version of global spectra are families $\{E_G\}_G$ of genuine G-spectra, in the sense of [31], for all compact Lie groups, equipped with transition morphisms $\xi_\alpha : \alpha^*(E_G) \longrightarrow E_H$ in the homotopy category of H-spectra. These data must satisfy some conditions, including that ξ_α is an isomorphism whenever α is the inclusion of a closed subgroup. While these objects certainly qualify as 'coherent families of equivariant spectra', the coherence is rather crude in that the transition morphism ξ_α only lives *in the homotopy category*. This definition captures the natural examples, but today's higher categorical perspective immediately craves for a refined definition where the transition maps ξ_α are actual morphisms (and not just homotopy classes) and the equalities in the Lewis–May definition are witnessed by coherence homotopies.

It took some time before a satisfactory and fully higher-categorical definition of global spectra as coherent families of equivariant spectra was provided by Linskens, Nardin and Pol [32]. In more technical terms, they identify the ∞-category of global spectra with a partially lax limit of a specific diagram of stable ∞-categories. That diagram, very roughly speaking, sends every compact Lie group G to the ∞-category $\mathcal{S}p_G$ of genuine G-spectra, and every continuous homomorphism $\alpha : K \longrightarrow G$ to the restriction functor $\alpha^* : \mathcal{S}p_G \longrightarrow \mathcal{S}p_K$. The marked morphisms are the *injective* homomorphisms α, and the partial laxness implements the design feature that the transition morphisms $\xi_\alpha : \alpha^*(E_G) \longrightarrow E_H$ of a global spectrum are equivalences whenever α is injective.

Global spectra are genuine cohomology theories on orbifolds and stacks. Various parts of pure mathematics have come up with notions of geometric objects that locally look like the quotient of a smooth object by a group action, in a way that remembers information about the isotropy groups of the action; examples are orbifolds in differential topology, and stacks in algebraic geometry. Such 'stacky' objects can behave like smooth objects even if the underlying spaces have singularities. As for spaces, manifolds and schemes, cohomology theories are important invariants also for stacks and orbifolds, and examples such as ordinary cohomology or K-theory lend themselves to generalization. Orbispaces and global spaces are slightly different implementations of the same idea with spaces (or rather, homotopy types, or animae) instead of 'smooth' objects. By now, there is a plethora of different approaches to defining global spaces. Historically, the first model is that of Gepner–Henriques [20]; it is given by contravariant continuous functors on a specific model for the global indexing category Glo, using homotopy orbits by the conjugation action on group homomorphisms. The author's orthogonal space model [38, Chapter 1] is a different setup that is easy to connect to orthogonal spectra. The closest thing to realizing the slogan that global objects have 'compatible actions of all groups' is the model as spaces with an action of the universal compact Lie group [40]; while the universal compact Lie group is neither compact nor a group, it contains all compact Lie groups as subgroups, in a specific way. For finite groups, even small 1-categories can be used to model global spaces, see [39]. Linskens, Nardin and Pol show in [32, Theorem 6.17] that global spaces are 'coherent families of equivariant spaces' in much the same way as global spectra are coherent families of equivariant spectra. Clough, Cnossen and Linskens [12] make precise in which way global spaces 'are' the homotopy theory of smooth stacks: they identify the ∞-category of global spaces with homotopy invariant sheaves of animae on the site of separated differentiable stacks.

Special cases of orbifolds are 'global quotients', sometimes denoted $G\backslash\backslash M$, for example for a smooth action of a finite group G on a smooth manifold M. If $E = \{E_G\}$ is a global spectrum, the orbifold cohomology of $G\backslash\backslash M$ is supposed to be the G-equivariant E_G-cohomology of M. The partial-laxness condition in the coherent families picture of global spectra arises very naturally from the orbispace perspective: if H is a subgroup of G, and N is an H-manifold, then $H\backslash\backslash N$ and $G\backslash\backslash(G \times_H N)$ are different presentations of the same orbifold; so the E_H-cohomology of N better agree with the E_G-cohomology of $G \times_H N$, which forces the underlying H-spectrum of E_G to be equivalent to E_H. In other words: the collection of equivariant spectra $\{E_G\}$ needs to be consistent under restriction to subgroups.

Global spectra are spectral global Mackey functors. The equivariant homotopy groups of global spectra come with a rich algebraic structure, namely that of a *global Mackey functor*. Restricted to finite groups, the notion of global Mackey functors has previously featured in various places in algebra and representation theory, sometimes under alternative names such as 'inflation functors' [48] or 'biset functors' [8].

Since I want to illustrate the computational power of global Mackey functors, I take the time to spell out a definition. My preferred definition is actually as additive functors from the *global Burnside category* of [38, Construction 4.2.1] to abelian groups. But there is an equivalent and more down-to-earth reformulation in terms of generating operations and relations that is more in tune with the narrative of this expository article.

Definition 3.1. A *global Mackey functor* M consists of:

(a) an *abelian group* $M(G)$ for every compact Lie group G,

(b) a *restriction homomorphism* $\alpha^* : M(G) \longrightarrow M(K)$ for every continuous group homomorphism $\alpha : K \longrightarrow G$,

(c) a *transfer homomorphism* $\mathrm{tr}_H^G : M(H) \longrightarrow M(G)$ for every closed subgroup H of G.

These data must satisfy the following relations:

(i) restriction is contravariantly functorial: $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$

(ii) transfers are covariantly functorial: $\mathrm{tr}_H^G \circ \mathrm{tr}_K^H = \mathrm{tr}_K^G$

(iii) inner automorphisms act as the identity

(iv) transfers commute with restriction along epimorphism

(v) $\mathrm{tr}_H^G = 0$ whenever $\dim(N_G H) > \dim(H)$,

(vi) the *double coset formula*:

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G = \sum_{[M] \in K\backslash G/H} \chi^\sharp(M) \cdot \mathrm{tr}_{K\cap {}^g H}^K \circ c_g^* \circ \mathrm{res}_{K^g \cap H}^H \tag{3.1}$$

Readers familiar with G-Mackey functors for finite groups will immediately realize that every global Mackey functor has underlying G-Mackey functors, by simply forgetting a lot of structure. Conversely, a generic G-Mackey functor does *not* extend to a global one. The simplest obstruction is that in a global functor, the presence of 'inflations' (i.e., restriction along surjective group homomorphisms) forces the restriction map $\mathrm{res}_{\{1\}}^G : M(G) \longrightarrow M(\{1\})$ to be a split epimorphism. For a general G-Mackey functor, this restriction map need not even be surjective.

Considering Mackey functors defined on compact Lie groups (as opposed to just finite groups) makes for a much richer story. While most of the structure behaves as for finite groups, the generalization exhibits some new features, most notably properties (v) and (vi) above. The double coset formula (vi) is the most complicated item in the definition, and for full details I refer to [38, Section 3.4]. When H is a subgroup of G of the same dimension (and hence of finite index), the double coset formula (vi) simplifies a lot. In this case the double coset space $K\backslash G/H$ is a discrete finite set, all the manifold components M are just points, all the internal Euler characteristics $\chi^\sharp(M)$ equal 1, and the double coset formula has the same form as in the realm of finite groups. But the double coset formula is particularly powerful when H and K have strictly smaller dimensions than G. Then the double coset space $K\backslash G/H$ is typically not discrete, but rather stratified by conjugacy classes of subgroups of K, and one needs to understand the stratification in concrete terms. I have gotten a lot of mileage out of special cases of the double coset formula where G is an orthogonal or unitary group, and K and H are smaller orthogonal or unitary groups, or block subgroups.

In the 'coherent family' picture for global spectra, half of the algebraic structure of the associated global Mackey functors – namely the restriction homomorphisms – are the effect on homotopy groups of the transition maps $\xi_\alpha : \alpha^*(E_G) \longrightarrow E_H$. The other half – the transfer maps – arise from the 'genuineness' of the equivariant constituent spectra in the coherent global family. But not only does a global spectrum E give rise to a global Mackey functor $\underline{\pi}_0(E) = \{\pi_0^G(E), \alpha^*, \mathrm{tr}_H^G\}$, this gadget is the full natural algebraic structure on the 0th equivariant homotopy groups. Moreover, the global stable homotopy category comes with a preferred t-structure whose heart is the abelian category of global Mackey functors, see [38, Theorem 4.4.9].

For $\mathcal{F}in$-global spectra, i.e., global spectra indexed only on finite groups (as opposed to general compact Lie groups), there is an even tighter connection to Mackey functors: $\mathcal{F}in$-global spectra are equivalent to spectral global Mackey functors. The precise implementation of this slogan goes as follows. In [4], Barwick introduces a very general higher-categorical framework of spectral Mackey functors. Given an ∞-category with pullbacks and specified subcategories to parameterize 'forward' and 'backwards' morphism (subject, of course, to axioms), he defines spectral Mackey functors as additive functors from an ∞-category of spans to the ∞-category of spectra. When applied to finite G-sets for a finite group G, and allowing forward and backward functoriality for all morphisms, this yields a model for genuine G-spectra. Replacing finite G-sets by finite groupoids yields a model for $\mathcal{F}in$-global stable homotopy theory as spectral Mackey functors; here arbitrary functors between finite groupoids are allowed for backward functoriality, but only faithful functors give forward functoriality. Lenz [29, Theorem A] constructed an equivalence between this ∞-category of spectral Mackey functors on spans of finite groupoids, and the ∞-category underlying Hausmann's symmetric spectrum model [22] for $\mathcal{F}in$-global spectra. Truncating this higher categorical equivalence down to the level of 1-categories recovers the structure of global Mackey functors on the equivariant homotopy groups of global spectra.

After this discussion of the general nature of global spectra, I would like to mention some important global (co)homology theories, their representing global spectra, and the associated global Mackey functors.

- Bredon equivariant (co)homology with integral coefficients is represented by the global Eilenberg–MacLane spectrum $H\underline{\mathbb{Z}}$. The associated global Mackey functor is 'constant', i.e., its value at every compact Lie group is $\mathbb{Z}$, and all restriction homomorphisms α^* are the identity. To satisfy the double coset formula, the transfers cannot generally be the identity; instead, tr_H^G is multiplication by the Euler charac-

teristic of the homogeneous space G/H. In particular, if H has the same dimension as G, then the transfer is multiplication by the index $[G : H]$.
More generally, every global Mackey functor has an associated global Eilenberg–MacLane spectrum. This construction implements an equivalence between the abelian category of global Mackey functors and the heart of the preferred t-structure on the global stable category, see [38, Theorem 4.4.9].

- Equivariant K-theory is represented by a global spectrum **KU**. This spectrum comes with a highly structured commutative multiplication ('ultra-commutative') that not only yields products, but also power operations and multiplicative norms on the equivariant cohomology theories. The Atiyah–Bott–Shapiro orientation is realized by a morphism of ultra-commutative ring spectra [27, Section 6]. The associated global Mackey functor sends a compact Lie group G to its representation ring $R(G)$, the Grothendieck group of finite-dimensional complex G-representations, with product induced by tensor product of representations. The restriction maps $\alpha^* : R(G) \longrightarrow R(K)$ are induced by restriction of representations along a continuous homomorphism $\alpha : K \longrightarrow G$. The transfer map $\mathrm{tr}_H^G : R(H) \longrightarrow R(G)$ along a closed subgroup inclusion $H \leq G$ is Segal's *smooth induction* [43, §2]. Whenever H has finite index in G, this sends an H-representation to the induced G-representation; in general, induction may send actual representations to virtual representations.
- The global sphere spectrum $\mathbb{S}$ is the coherent family of G-equivariant sphere spectra, with $\mathbb{S}_G$ representing G-equivariant cohomotopy. The associated Mackey functor sends G to the G-equivariant 0-stem $\pi_0^G(\mathbb{S})$, defined as the colimit over all G-representations V of homotopy classes of G-equivariant selfmaps of S^V. When G is finite, then $\pi_0^G(\mathbb{S})$ is naturally isomorphic to the Burnside ring $A(G)$, the Grothendieck group of finite G-sets. The unit morphism $\mathbb{S} \longrightarrow \mathbf{KU}$ of the global K-theory spectrum refines the morphism of global Mackey functors whose value $A(G) \longrightarrow R(G)$ sends a finite G-set to the associated permutation representation. In general, $\pi_0^G(\mathbb{S})$ is a free abelian group with basis the transfers $\mathrm{tr}_H^G(1)$ as H runs over all conjugacy classes of closed subgroups with finite Weyl group in G.
- The family of Borel equivariant spectra associated to a non-equivariant spectrum is global. The construction is implemented by the lax symmetric monoidal functor $b : Sp \longrightarrow Sp_{\mathrm{gl}}$ from non-equivariant spectra to global spectra that is right adjoint to the forgetful functor, compare [38, Construction 4.5.21].

The two equivariant versions $\mathbf{mO}_G$ and $\mathbf{MO}_G$ of the classical Thom spectrum are both underlying global spectra, denoted **mO** and **MO**, respectively. I return to these and their complex cousins **mU** and **MU** in more detail in Section 4.

The extra global structure can be, and has been, put to good use. The following results are examples where globally-equivariant structure was used in an essential way to resolve a problem in equivariant algebraic topology.

- In [37], I use the global structure to calculate the equivariant 0-stems of the G-equivariant spectra made from symmetric products of representations spheres.
- In [23], Hausmann proves a conjecture of Greenlees: for abelian compact Lie groups A, the homotopical equivariant bordism ring $\mathbf{MU}_A^*$ carries the universal A-equivariant formal group.
- In [41], I prove a splitting theorem for the values of global Mackey functors at orthogonal, unitary and symplectic groups; an immediate consequence is the regularity of certain $U(n)$-equivariant Euler classes in homotopical equivariant bordism. This triggered the construction of $\mathbf{MU}_G$-Chern classes with enough regularity properties [42]. I will elaborate on these applications in the final Section 4.
- In [28], La Vecchia proves the Greenlees–May conjecture regarding a completion theorem for equivariant $\mathbf{MU}_G$-module spectra for general compact Lie groups, making essential use of the splitting from [41].
- In [24], Hausmann and the author establish a universal characterization of geometric bordism for manifolds with commuting involutions, also known as equivariant bordism for elementary abelian 2-groups.

4. Case study: equivariant bordism

In this final section I want to illustrate by two examples how the global structure, specifically the restriction, inflation and transfer operations, and the double coset formula, work in practice. Both examples involve *homotopical equivariant bordism* originally introduced by tom Dieck [46]. Before diving into the new results, I will provide some more context.

The non-equivariant bordism ring is well understood. In his celebrated paper on the subject, Thom [45] showed that the bordism ring $\mathcal{N}_*$ of smooth closed manifolds is a polynomial $\mathbb{F}_2$-algebra on infinitely many generators x_i of dimension $i \geq 2$, for $i+1$ *not* a power of 2. Thom also showed that for even i, the bordism classes of the real projective spaces $\mathbb{R}P^i$ can be taken as polynomial generators. Only slightly later, Dold [17] exhibited specific manifolds (now called the 'Dold manifolds') that can serve as the odd dimensional polynomial generators. There is also a convenient necessary and sufficient criterion to decide when two smooth closed manifolds are bordant: all of their Stiefel–Whitney characteristic numbers need to coincide.

The serious study of equivariant bordism was initiated by the work [14] of Conner and Floyd. For a compact Lie group G, let $\mathcal{N}_n^G$ denote the group, under disjoint union, of n-dimensional smooth closed G-manifolds, taken up to equivariant bordism. Then for varying n, the graded abelian groups $\mathcal{N}_*^G$ have a ring structure by product of equivariant manifolds. Endowing non-equivariant manifolds with trivial G-actions makes $\mathcal{N}_*^G$ into an algebra over the non-equivariant bordism ring $\mathcal{N}_*$. Conner and Floyd carefully studied the bordism ring of manifolds with involution $\mathcal{N}_*^{C_2}$, i.e., the equivariant bordism ring for the group $G = C_2 = \{\pm 1\}$ with two elements. Among other things, Conner–Floyd showed that $\mathcal{N}_*^{C_2}$ is a free module of infinite rank (but of finite type) over the non-equivariant bordism ring $\mathcal{N}_*$. Alexander [3] described an explicit geometric basis constructed inductively from the $\mathbb{R}P^n$'s with involution $[x_0 : x_1 : \ldots : x_n] \mapsto [-x_0 : x_1 : \ldots : x_n]$, along with some partial information about the ring structure.

Not only do the bordism rings $\mathcal{N}_*^G$ exist 'uniformly' for all compact Lie groups, they also admit global functoriality. Indeed, a continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups is automatically smooth, and hence restriction of actions along α yields a *restriction homomorphism* $\alpha^* : \mathcal{N}_n^G \longrightarrow \mathcal{N}_n^K$. If H is a closed subgroup of G, and M a smooth closed H-manifold, then the induced G-space $G \times_H M$ has a unique smooth structure for which the projection $G \times M \longrightarrow G \times_H M$ is a submersion. Moreover, this operation passes to a *transfer homomorphism* (or 'induction homomorphism')

$$\mathrm{tr}_H^G \ : \ \mathcal{N}_n^H \ \longrightarrow \ \mathcal{N}_{n+\dim(G/H)}^G$$

between the bordism groups. If H has a smaller dimension than G, there is an increase in dimension by $\dim(G/H) = \dim(G) - \dim(H)$. This essentially (but not quite) makes the geometric bordism rings into a global Mackey functor.

As I already emphasized earlier, classical cohomology theories and spectra have many equivariant and global refinements. And sometimes more than one such global form is interesting; equivariant bordism is a prime example of this phenomenon. The complex (or unitary) bordism spectrum MU plays several important roles in stable homotopy theory:

(a) MU represents the bordism homology theory of smooth manifolds with stably almost complex structures;

(b) MU is the universal complex oriented cohomology theory, i.e., a multiplicative cohomology theory endowed with natural Thom isomorphisms; and

(c) by Quillen's celebrated theorem [35], the coefficient ring MU^* together with the formal group law arising from the tautological complex orientation, is an initial formal group law.

One might then desire, for every compact Lie group G, an equivariant theory that enjoys analogous properties. However, when G is nontrivial, properties (a) and (b) cannot simultaneously be satisfied. Indeed, manifolds have non-negative dimensions, so geometrically defined bordism theories are necessarily concentrated in non-negative degrees. However, equivariant complex orientations in particular provide Thom classes for G-representations; and for representations with trivial fixed points, the associated Euler classes are nontrivial homotopy classes in *negative* degrees.

In my opinion, the fact that equivariant theories cannot simultaneously be connective and complex oriented is a global feature, and not a bug. For example, there are two very interesting global ring spectra **mU** and **MU** that refine the non-equivariant theory, with **mU** being equivariantly connective, and **MU** being equivariantly complex oriented. The fact that the geometric-to-homotopical morphism of global spectra $\mathbf{mU} \longrightarrow \mathbf{MU}$ is *not* an equivalence at nontrivial groups can also be seen as an incarnation of the failure of equivariant transversality.

Tom Dieck's theory $\mathbf{MU}_G^*$ is manufactured from equivariant Thom spaces of universal bundles over equivariant Grassmannians; it is the universal G-equivariantly complex oriented cohomology theory. The coefficient rings $\mathbf{MU}_G^*$ are reasonably well-understood for *abelian* G; but, except possibly for $G = \{\pm 1\}$, not in the sense of a useful presentation by generators and relations. For abelian G, the graded ring $\mathbf{MU}_G^*$ is concentrated in even degrees and free as a module over the non-equivariant cobordism ring MU^*; the bundling homomorphism $\mathbf{MU}_G^* \longrightarrow MU^*(BG)$ is completion at the augmentation ideal; and the G-equivariant formal group law over $\mathbf{MU}_G^*$ arising from the tautological complex orientation is initial [23]. For nonabelian compact Lie groups, however, the equivariant bordism rings $\mathbf{MU}_G^*$ are still largely mysterious.

As already mentioned, the theories $\{\mathbf{MU}_G^*\}_G$ assemble into a global ring spectrum **MU**, and I now want to illustrate how the global structure has been used to elucidate the structure of the rings $\mathbf{MU}_G^*$ for unitary groups. A key player in this game is the *Euler class*

$$e_n \in \mathbf{MU}_{U(n)}^{2n}$$

of the tautological $U(n)$-representation on $\mathbb{C}^n$, already defined by tom Dieck [46, page 347] when he introduced the theory. This Euler class participates in a classical long exact sequence (Gysin sequence):

$$\ldots \longrightarrow \mathbf{MU}_{U(n)}^{*-2n} \xrightarrow{e_n \cdot} \mathbf{MU}_{U(n)}^{*} \xrightarrow{\mathrm{res}_{U(n-1)}^{U(n)}} \mathbf{MU}_{U(n-1)}^{*} \longrightarrow \ldots$$

The main result of [41] says that the restriction map $\mathrm{res}_{U(n-1)}^{U(n)}$ is surjective; thus the long exact sequence decomposes into short exact sequences. Hence the Euler class e_n is *regular*, i.e., multiplication by e_n is injective.

The surjectivity of the restriction map $\mathrm{res}_{U(n-1)}^{U(n)}$ is not specific to **MU**, and it works in the same way for every global spectrum, and even for every global Mackey functor. In fact, in [41] I use the global structure to exhibit a natural additive splitting. On the kernel of $\mathrm{res}^{U(n-1)}_{U(n-2)}$, the splitting is given by the composite

$$\mathbf{MU}^*_{U(n-1)} \xrightarrow{p^*} \mathbf{MU}^*_{U(n-1)\times U(1)} \xrightarrow{\mathrm{tr}^{U(n)}_{U(n-1)\times U(1)}} \mathbf{MU}^*_{U(n)} .$$

The first map is an inflation, i.e., restriction along the projection $p : U(n-1) \times U(1) \longrightarrow U(n-1)$ to the first factor; the second map is the transfer from the block matrix subgroup $U(n-1) \times U(1)$ to $U(n)$. So this splitting has a similar flavor as parabolic induction in the representation theory of reductive algebraic groups. In verifying that the construction really provides a splitting, one needs to confront expressions like

$$\mathrm{res}^{U(n)}_{U(n-1)} \circ \mathrm{tr}^{U(n)}_{U(n-1)\times U(1)} ,$$

a specific composite of a transfer and a restriction, and thus work out an instance of the double coset formula (3.1) where the double coset space is not discrete. In this particular situation, the double coset space is a closed interval, with the minimal stratification by its two end points and the interior.

The final illustration of the global structure is the construction of Chern classes in homotopical equivariant bordism. Chern classes are important characteristic classes for complex vector bundles that were originally introduced in singular cohomology. Conner and Floyd [15, Corollary 8.3] constructed Chern classes for complex vector bundles in complex cobordism, nowadays referred to as Conner–Floyd–Chern classes.

Popular constructions of Chern classes manufacture them from the Euler classes of a complex orientation via either the projective bundle theorem, or the 'splitting principle'. However, these tools are *not* available in the equivariant context. For example, the splitting theorem says that for complex oriented E, the homomorphism $E^*(BU(n)) \longrightarrow E^*(T)$ induced by the inclusion of the maximal torus $T = U(1)^n$ into $U(n)$ is injective. In homotopical equivariant bordism, the restriction homomorphism

$$\mathrm{res}^{U(n)}_T : \mathbf{MU}^*_{U(n)} \longrightarrow \mathbf{MU}^*_T$$

is however *not* injective for $n \geq 2$. So the traditional methods fail to produce Chern-like classes in homotopical equivariant bordism.

At this point, the global structure of **MU** comes to rescue. We start from the Euler class $e_k \in \mathbf{MU}^{2k}_{U(k)}$ of the tautological $U(k)$-representation on $\mathbb{C}^k$, inflate along the projection $p : U(k) \times U(n-k) \longrightarrow U(k)$, and then transfer up to $U(n)$: the k-Chern class is

$$c_k = \mathrm{tr}^{U(n)}_{U(k)\times U(n-k)}(p^*(e_k)) \quad \text{in } \mathbf{MU}^{2k}_{U(n)}.$$

These Chern classes satisfy the analogous formal properties as their classical counterparts, including the equivariant refinement of the Whitney sum formula. Since c_k is defined as a transfer, verifying these properties again involves a double coset formula

(3.1), this time with double coset space a simplex stratified by its faces. Despite the many formal similarities, there are crucial qualitative differences compared to Chern classes in complex oriented cohomology theories: The **MU**-Chern classes are not characterized by their restriction to the maximal torus, and some of them are zero-divisors.

With the Whitney sum formula and the regularity of [41] at hand, it is only a short way to the main result of [42]:

Theorem 4.1.

(i) *The Chern classes $c_n, c_{n-1}, \dots, c_1$ form a regular sequence in $\mathbf{MU}^*_{U(n)}$ that generates the augmentation ideal.*

(ii) *The completion of $\mathbf{MU}^*_{U(n)}$ at the augmentation ideal is a power series algebra over $\mathbf{MU}^*$ in the Chern classes, and tom Dieck's bundling map extends to an isomorphism*

$$(\mathbf{MU}^*_{U(n)})^{\wedge}_I \;\cong\; \mathbf{MU}^*(BU(n)).$$

I believe that the global techniques have further potential to illuminate the remaining mysteries surrounding homotopical equivariant bordism, such as the evenness conjecture, or the quest for a notion of equivariant formal group law for nonabelian compact Lie groups.

This ends my tour of global homotopy theory. To wrap up, some of the main points I was trying to convey are:

- global homotopy theory is the home of equivariant phenomena with 'universal symmetry';
- many interesting equivariant cohomology theories are global;
- recognizing cohomology theories as global provides rich algebraic structure;
- homotopical equivariant bordism witnesses the calculational impact of global equivariant structures.

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