HOCHSCHILD COHOMOLOGY OF THE CLUSTER-TILTED ALGEBRAS OF FINITE REPRESENTATION TYPE

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Abstract. We compute the Hochschild cohomology groups of the cluster-tilted algebras of finite representation type.

An important homological invariant of a finite-dimensional algebra Λ over a field K is its Hochschild cohomology, defined as the graded ring $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^{op} \otimes_K \Lambda}(\Lambda, \Lambda)$, see [14]. Even if Λ is given combinatorially as quiver with relations, it is not easy to explicitly determine the groups $\mathrm{HH}^i(\Lambda)$, and in many cases one needs a projective resolution of Λ as a bimodule over itself.

An interesting class of algebras consists of the cluster-tilted algebras introduced by Buan, Marsh and Reiten [7] as the endomorphism algebras of cluster-tilting objects in a cluster category. Cluster-tilted algebras of finite representation type were studied in [6], see also [10]. They can be described by quivers with relations where the quivers are obtained from orientations of ADE Dynkin diagrams by performing sequences of quiver mutations [13], and the defining relations consist of zero- and commutativity-relations that can be deduced from the quiver in an algorithmic way. From a homological viewpoint, cluster-tilted algebras are Gorenstein [15], but in general they are of infinite global dimension.

Previously, the first Hochschild cohomology group of a schurian cluster-tilted algebra was computed in terms of an underlying tilted algebra, see [1]. In this note we compute all the Hochschild cohomology groups $\mathrm{HH}^i(\Lambda)$ of a cluster-tilted algebra Λ of finite representation type in terms of its quiver.

In order to formulate our results, we encode the dimensions of $\mathrm{HH}^i(\Lambda)$ in a formal power series

$$h_\Lambda(z) = \sum_{i=0}^{\infty} \dim_K \mathrm{HH}^i(\Lambda) \cdot z^i - 1$$

and we define, for $n \geq 3$, the formal power series

$$f_n(z) = \frac{z}{1-z} - \frac{z^2\left(1 + \varepsilon_n(z+z^2) + z^3\right)}{1-z^{2n}}, \quad \text{where } \varepsilon_n = \begin{cases} 0 & \text{if } \operatorname{char} K \text{ divides } n-1, \\ 1 & \text{otherwise.} \end{cases}$$

The cluster-tilted algebras of Dynkin type A have been described as quivers with relations in [8, 9].

Theorem 1 (Dynkin type A)**.** *Let Λ be a cluster-tilted algebra of Dynkin type A and let t be the number of oriented 3-cycles in the quiver of Λ. Then $h_\Lambda(z) = tf_3(z)$.*

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The quivers in the mutation class of a Dynkin quiver of type D have been explicitly described in [19], where they were organized into four types. In the next theorem we use the terminology of [5, §1.6] concerning types and parameters.

Theorem 2 (Dynkin type D). *Let Λ be a cluster-tilted algebra of Dynkin type D.*

(I) *If Λ is of type I with parameters (s,t), then*
$$h_\Lambda(z) = tf_3(z).$$
(II) *If Λ is of type II with parameters (s_1,t_1,s_2,t_2), then*
$$h_\Lambda(z) = (1+t_1+t_2)f_3(z).$$
(III) *If Λ is of type III with parameters (s_1,t_1,s_2,t_2), then*
$$h_\Lambda(z) = f_4(z) + (t_1+t_2)f_3(z).$$
(IVa) *If Λ is of type IV and its quiver is an oriented cycle of length n, then*
$$h_\Lambda(z) = f_n(z).$$
(IVb) *If Λ is of type IV with parameters $\big((d_1,s_1,t_1),(d_2,s_2,t_2),\dots,(d_r,s_r,t_r)\big)$, then*
$$h_\Lambda(z) = f_n(z) + (t_1+t_2+\dots+t_r)f_3(z),$$
where $n = d_1+\dots+d_r+|\{1\le j\le r \,:\, d_j=1\}|$.

Remark. The cluster-tilted algebra corresponding to an oriented cycle is a truncated cycle algebra. The Hochschild cohomology of such algebras was considered by many authors, see [2, 12, 16, 20].

We define the *associated polynomial* of an algebra Λ as $\det(xC_\Lambda - C_\Lambda^T) \in \mathbb{Z}[x]$, where C_Λ denotes the Cartan matrix of Λ. The cluster-tilted algebras of Dynkin type E have been classified up to derived equivalence in [4], where it is shown that the associated polynomial is a complete derived invariant for these algebras.

Theorem 3 (Dynkin type E). *Let Λ be a cluster-tilted algebra of Dynkin type E. Then $h_\Lambda(z)$ is determined by the associated polynomial of Λ according to Table 1.*

We list a few consequences of these results. The first states that cluster-tilted algebras of finite representation type are rigid.

Corollary 1. $\mathrm{HH}^2(\Lambda)=0$ *for any cluster-tilted algebra of finite representation type Λ.*

Another consequence is that the Hochschild cohomology groups of a cluster-tilted algebra of finite representation type are completely determined by its first Hochschild cohomology and the determinant of its Cartan matrix. These determinants were computed in [4, 5, 8].

Corollary 2. *Let Λ be a cluster-tilted algebra of finite representation type. Then*
$$h_\Lambda(z) = f_n(z) + tf_3(z)$$
where $t = \dim \mathrm{HH}^1(\Lambda) - 1$ and $n = 1 + 2^{-t}\det C_\Lambda$.

In particular, for two cluster-tilted algebras Λ, Λ' of finite representation type the following conditions are equivalent:

(i) $\mathrm{HH}^1(\Lambda) \simeq \mathrm{HH}^1(\Lambda')$ *and* $\det C_\Lambda = \det C_{\Lambda'}$*;*

Associated polynomial	$\mathbf{h_\Lambda(z)}$
$x^6 - x^5 + x^3 - x + 1$	0
$2(x^6 - 2x^4 + 4x^3 - 2x^2 + 1)$	$f_3(z)$
$2(x^6 - x^4 + 2x^3 - x^2 + 1)$	$f_3(z)$
$3(x^6 + x^3 + 1)$	$f_4(z)$
$4(x^6 + x^4 + x^2 + 1)$	$f_5(z)$
$4(x^6 + x^5 - x^4 + 2x^3 - x^2 + x + 1)$	$2f_3(z)$
$x^7 - x^6 + x^4 - x^3 + x - 1$	0
$2(x^7 - 2x^5 + 4x^4 - 4x^3 + 2x^2 - 1)$	$f_3(z)$
$2(x^7 - x^5 + x^4 - x^3 + x^2 - 1)$	$f_3(z)$
$2(x^7 - x^5 + 2x^4 - 2x^3 + x^2 - 1)$	$f_3(z)$
$3(x^7 - 1)$	$f_4(z)$
$4(x^7 + x^5 - 2x^4 + 2x^3 - x^2 - 1)$	$f_5(z)$
$4(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	$f_5(z)$
$4(x^7 + x^6 - 2x^5 + 2x^4 - 2x^3 + 2x^2 - x - 1)$	$2f_3(z)$
$4(x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1)$	$2f_3(z)$
$4(x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x - 1)$	$2f_3(z)$
$5(x^7 + x^5 - x^4 + x^3 - x^2 - 1)$	$f_6(z)$
$6(x^7 + x^5 - x^2 - 1)$	$f_7(z)$
$6(x^7 + x^6 - x^4 + x^3 - x - 1)$	$f_4(z) + f_3(z)$
$8(x^7 + x^6 + x^5 - x^4 + x^3 - x^2 - x - 1)$	$f_5(z) + f_3(z)$
$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$	0
$2(x^8 - 2x^6 + 4x^5 - 4x^4 + 4x^3 - 2x^2 + 1)$	$f_3(z)$
$2(x^8 - x^6 + x^5 + x^3 - x^2 + 1)$	$f_3(z)$
$2(x^8 - x^6 + 2x^5 - 2x^4 + 2x^3 - x^2 + 1)$	$f_3(z)$
$3(x^8 + x^4 + 1)$	$f_4(z)$
$4(x^8 + x^6 - 2x^5 + 4x^4 - 2x^3 + x^2 + 1)$	$f_5(z)$
$4(x^8 + x^6 - x^5 + 2x^4 - x^3 + x^2 + 1)$	$f_5(z)$
$4(x^8 + x^7 - 2x^6 + 2x^5 + 2x^3 - 2x^2 + x + 1)$	$2f_3(z)$
$4(x^8 + x^7 - x^6 + 2x^4 - x^2 + x + 1)$	$2f_3(z)$
$4(x^8 + x^7 - x^6 + x^5 + x^3 - x^2 + x + 1)$	$2f_3(z)$
$5(x^8 + x^6 + x^4 + x^2 + 1)$	$f_6(z)$
$6(x^8 + x^6 + x^5 + x^3 + x^2 + 1)$	$f_7(z)$
$6(x^8 + x^7 + 2x^4 + x + 1)$	$f_4(z) + f_3(z)$
$8(x^8 + x^7 + x^6 + 2x^4 + x^2 + x + 1)$	$f_5(z) + f_3(z)$
$8(x^8 + 2x^7 + 2x^4 + 2x + 1)$	$3f_3(z)$

Table 1. The Hochschild cohomology groups as functions of the associated polynomial for cluster-tilted algebras of types E_6, E_7, E_8.

(ii) $\mathrm{HH}^i(\Lambda) \simeq \mathrm{HH}^i(\Lambda')$ *for all* $i \geq 0$.

Remark. Fixing the number of simples, we see that the Hochschild cohomology is a complete derived invariant for cluster-tilted algebras of Dynkin type A. This is no longer true in Dynkin types D and E.

Our results are based on several ingredients. The first is the explicit knowledge of the quivers of the cluster-tilted algebras in question [4, 8, 9, 19]. The second ingredient is a reduction technique based on the long exact sequences of [11, 14, 17] allowing one to decompose the problem of computing the Hochschild cohomology of a cluster-tilted algebra of finite representation type into smaller problems involving simpler cluster-tilted algebras. However, some of these simpler algebras are not monomial, hence the projective resolution given in [3] is not always applicable. In order to overcome this difficulty, we use the invariance of Hochschild cohomology under derived equivalence [14, 18] and replace these algebras by derived equivalent ones whose quivers are oriented cycles and their defining relations consist of only zero-relations of varying lengths. In general, these monomial algebras are not cluster-tilted anymore. Finally, by applying another reduction technique we are able to shorten the cycles and show that the Hochschild cohomology of these monomial algebras is isomorphic to that of certain truncated cycle algebras. The Hochschild cohomology of the latter algebras was computed by several authors [2, 12, 16, 20].

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