HOMEWORK #3 IN ALGEBRAIC STRUCTURES 2

Problem 3.1. Prove Lemma 3.4.

Problem 3.2. Prove Lemma 3.6.

Problem 3.3. Show that the following polynomials in $\mathbb{Q}[t]$ are irreducible:
a) $f(t) = 5t^4 - 7t + 7$,
b) $f(t) = t^{p-1} + t^{p-2} + ... + t + 1$ where p is a prime number.
Hint: Apply the Eisenstein criteria to $g(t) := f(t+1)$ for the prime p.

Problem 3.4. Let $K := \mathbb{F}_p(x, y)$. Show that
a) the polynomial $t^p - x \in K[t]$ is irreducible,
Let L be the field obtained from K by adjoining a root of the polynomial $t^p - x$.
b) the polynomial $t^p - y \in L[t]$ is irreducible,
Let M be the field obtained from L by adjoining a root of the polynomial $t^p - y$.
c) for any $m \in M$ we have $m^p \in K$
d) the extension $M \supset K$ is not elementary.

Problem 3.5. Let K be field such that every element of K is a square. Show that
a) if $\mathrm{ch}(K) \neq 2$ then any quadratic equation has a solution (in K)
b) if $\mathrm{ch}(K) = 2$ then any quadratic equation has a solution if any equation of the form

$$t^2 + t = a, a \in K$$

has a solution.

Problem 3.6. a) How many non-isomorphic quadratic extensions of $\mathbb{F}_5$ exist?
b) let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}), \alpha = \sqrt{2} + \sqrt{3} \in L$. Show that $L = \mathbb{Q}(\alpha)$

Quotient rings Let A be a commutative ring, $I \subset A$ an ideal. We define an equivalence relation on A by saying $a \equiv b$ if $a - b \in I$ and denote by A/I the corresponding set of equivalence classes. We denote by $a \to \bar{a}$ the map $A \to A/I$ assigning to any $a \in A$ the equivalence class $a + I \in A/I$.

Problem 3.7. Show that
a) if $a \equiv a', b \equiv b'$ then $a + b \equiv a' + b'$ and $ab \equiv a'b'$,

b) there exists operations $+ : A/I \times A/I \to A/I$ and $\times : A/I \times A/I \to A/I$ such that for any $a, b \in A$ we have

$$\overline{a+b} = \bar{a} + \bar{b}, \overline{a \times b} = \bar{a}\bar{b}$$

c) the set A/I with operations $+ : A/I \times A/I \to A/I$ and $\times : A/I \times A/I \to A/I$, unit $= \bar{1}$ and zero $= \bar{0}$ has a structure of a commutative ring.

Problem 3.8. Show that:

a) for any polynomial $p(t) \in K[t]$ we have $\dim_K K[t]/(p(t)) = \deg p(t)$.

b) for any irreducible polynomial $p(t) \in K[t]$ the ideal $(p(t)) \subset K[t]$ is maximal.